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FRANKLIN MATHEMATICAL SERIES

ELEMENTARY

TRIGONOMETRY

PLANE AND SPHERICAL

BY

Pliny

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TAINTOR BROTHERS & CO
NEW YORK AND CHICAGO
BOSTON: WILLIAM WARE & CO

1889.

Math 8012.89.2

NOV 25 1889

E. P. Seaver

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PREFACE.

THIS book is intended to serve two distinct though not inconsistent purposes: first, to furnish a very brief practical course for those schools in which the study can claim but a little time; and, secondly, to present the whole subject of trigonometric functions in the manner in which the student of higher mathematics needs to have learned it.

The first purpose will be served by the first three chapters (pp. 1-65) of the book. In this part the functions of an acute angle only are defined and used. Right triangles are solved by direct reference to the definitions of functions, and oblique triangles by dividing them into right triangles. Not even a knowledge of logarithms is necessary for reading this part of the book, all the methods being illustrated by examples solved with three-place natural functions, a table of which is given on page 9. The application of logarithms is treated as a special matter in separate paragraphs. Among the exercises in the first three chapters occur various applications of trigonometry to the calculation of heights and distances.

The second purpose has governed in the preparation of the next four chapters (pp. 66-160). Here, after the generalized conception of the angle has been introduced, together with some notions of positive and negative lines and of rectangular coördinates, the trigonometric functions are defined as ratios of coördinates. The advantage of this is seen in laying the foundation for the general proof of the formulas for the sine and the cosine of the sum and of the difference of two angles. The usual method of proving these formulas by a geometrical figure in the first quadrant, the validity of the proof for figures that might be drawn in the other quadrants being assumed, seems, from a teacher's point of view, unsatisfactory; partly because fundamental formulas like these should be solidly established, and partly because this is the first important instance the mathematical student meets where a proposition proved for the first quadrant does hold for the others. Whatever he may afterwards allow himself to assume, he should in this instance at least carefully explore the whole ground of geometric configuration. These considerations have suggested the unusually thorough treatment of this topic in Chapter VI, and the introduction of Chapter V, on projections, to furnish a foundation for this treatment. In Chapter VII are treated the important properties of plane triangles. From these are deduced some methods for logarithmic solution which are simpler than those given earlier. When this chapter is to be studied, it may be found well to postpone the exercises near the end of Chapter III until advantage can be taken of these simpler methods.

In spherical trigonometry, Chapters VIII – XI, it has been found simpler and better to cast aside the usual limitation of the results of solution to such sides and angles as do not exceed 180°, and to accept and interpret all results. One step more removes a similar limitation from the data; and thus we have the general spherical triangle, treated in the last chapter.

Among the methods proposed for the solution of spherical triangles, attention may be called to the elegant ones furnished by the Gaussian equations. Notwithstanding the ambiguity attaching to the square roots of which these equations are composed, the results present but little difficulty of interpretation, as the examples in the last chapter show. In case, however, of any perplexity, the simple and direct method of solution based on the fundamental equations can be employed to remove it. By using three-place natural functions, a rough approximation can be made, which will show the general shape of the required triangle, and, except in extreme cases, give the values of the required parts within a fraction of a degree.

The method of solution which consists in the introduction of an auxiliary angle for the purpose of adapting the fundamental formulas to logarithmic computation is not added to the others, because it is step for step equivalent to the method which divides the triangle into two spherical right triangles by means of a perpendicular; and the latter method has the advantage of a direct geometrical interpretation.

One chapter on the development of trigonometric functions into series, De Moivre's theorem, and other such matters formed a part of the original plan of the book; but the other chapters have so outgrown intended limits that this must, for the present at least, be dropped.

E. P. S.

Boston, May, 1889.

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ELEMENTARY TRIGONOMETRY.

CHAPTER I.

TRIGONOMETRIC FUNCTIONS.

TRIGONOMETRY treats of the numerical relations between the sides and the angles of triangles.

Plane trigonometry treats of plane triangles, and spherical trigonometry of spherical triangles.

2. We know, from geometry, that there are in any plane triangle six parts, three sides and three angles, which are so related to one another that when any three parts are given, provided one be a side, the other three parts can be found. There are simple geometric constructions by which this can be done. To make one of these constructions is to solve a triangle by geometry. The process is partly drawing and partly measurement.

But solving a triangle by trigonometry is wholly a process of numerical computation. The relations between the given and the required parts being expressed by formulas, the numerical values of the given parts are then substituted for their symbols in these formulas, and the required parts are computed. A geometrical figure may be drawn for illustration, but this is no necessary part of the solution.

3. The sides of a plane triangle having been measured, if the length of each side be divided by that of each of the other two, the six quotients will be the numerical values of the six ratios which the sides of the triangle bear to one another. If another triangle, having its angles equal to those of the first triangle, be treated in the same way, the six ratios will have the same values respectively as the six ratios of the first triangle; because triangles which are mutually equi-angular have their corresponding sides proportional. Conversely, if the six ratios of the second triangle were given as equal to the six ratios of the first, we could infer that the second triangle had angles equal to those of the first; because triangles having their corresponding sides proportional are mutually equi-angular.

Thus we reach the general conclusion that *all triangles having the same set of angles have also the same set of values for the six ratios of the sides;* and, conversely, *all triangles having the same set of values for the six ratios of the sides have also the same set of angles.*

4. To find the ratios from a given set of angles, or to find the angles from a given set of ratios, is geometrically a direct and simple process. If the angles are given, we draw any triangle with the given angles, measure the sides and divide their lengths by one another. This gives the ratios. If the ratios are given, we draw one line of any convenient length, and two others of such lengths that they bear to the first line the proper given ratios. The proper given ratios for this purpose are any two that are not reciprocals of each other. With the three lines thus found we construct a triangle and measure the angles. This gives the angles.

5. In mathematics the word *function* is used to denote a quantity whose value is dependent upon the values of one or more other quantities. Thus, the area of a circle is a function of the radius; the volume of a sphere is a function of the radius; the area of a parallelogram is a function of the base and altitude; the distance passed through by a moving body is a function of the rate and time; the amount of work done is a function of the force and time; and so on.

From what has been said above, it is clear that *the ratios of the sides of a triangle are functions of the angles,* and that *the angles of a triangle are functions of the ratios of the sides.* Upon this general relation between the sides and the angles of triangles is based the whole of trigonometry.

THE FUNCTIONS OF AN ACUTE ANGLE DEFINED.

6. Let A be any acute angle. From any point **B** in one of its sides draw a line **BC** perpendicular to the other side, thus forming the right triangle **ACB**. If the sides of this triangle be divided by one another, the six resulting ratios (which would have the same values in all other right triangles having A for an acute angle), are defined as functions of the angle A thus:

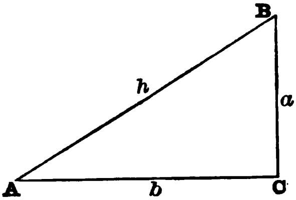

Fig. 1.

$$\text{The } \textit{sine} \text{ of } A = \frac{\text{the opposite side}}{\text{the hypothenuse}} = \frac{\mathrm{CB}}{\mathrm{AB}}$$

$$\text{The } \textit{tangent} \text{ of } A = \frac{\text{the opposite side}}{\text{the adjacent side}} = \frac{\mathrm{CB}}{\mathrm{AC}}$$

$$\text{The } \textit{secant} \text{ of } A = \frac{\text{the hypothenuse}}{\text{the adjacent side}} = \frac{\mathrm{AB}}{\mathrm{AC}}$$

$$\text{The } \textit{cosine} \text{ of } A = \frac{\text{the adjacent side}}{\text{the hypothenuse}} = \frac{\mathbf{AC}}{\mathbf{AB}}$$

$$\text{The } \textit{cotangent} \text{ of } A = \frac{\text{the adjacent side}}{\text{the opposite side}} = \frac{\mathbf{AC}}{\mathbf{CB}}$$

$$\text{The } \textit{cosecant} \text{ of } A = \frac{\text{the hypothenuse}}{\text{the opposite side}} = \frac{\mathbf{AB}}{\mathbf{CB}}$$

7. The same six ratios may be defined as functions of the angle B, the complement of A, as follows:

$$\text{The } \textit{sine} \text{ of } B = \frac{\text{the opposite side}}{\text{the hypothenuse}} = \frac{\mathbf{AC}}{\mathbf{AB}}$$

$$\text{The } \textit{tangent} \text{ of } B = \frac{\text{the opposite side}}{\text{the adjacent side}} = \frac{\mathbf{AC}}{\mathbf{CB}}$$

and so on.

8. Denoting the length of the hypothenuse by h, that of the side opposite A by a, and that of the side opposite B by b, we write, the *sine* of $A = \frac{a}{h}$, the *tangent* of $A = \frac{a}{b}$, and so on.

The names of the functions are abbreviated as in the following

TABLE OF DEFINITIONS.

$$\left.\begin{array}{ll} \sin A = \frac{a}{h} = \cos B & \cos A = \frac{b}{h} = \sin B \\ \tan A = \frac{a}{b} = \cot B & \cot A = \frac{b}{a} = \tan B \\ \sec A = \frac{h}{b} = \csc B & \csc A = \frac{h}{a} = \sec B \end{array}\right\} \quad . \; . \; [1]$$

THE FUNCTIONS OF COMPLEMENTARY ANGLES.

9. Since the two acute angles of a right-triangle are complements of each other, that is,

$$B = 90° - A, \text{ and } A = 90° - B,$$

we learn from [1][*] that the cosine, the cotangent, and the cosecant of an angle are respectively the sine, the tangent, and the secant of the angle's complement; also that the sine, the tangent, and the secant of an angle are respectively the cosine, the cotangent, and the cosecant of the angle's complement. Thus,

$$\left.\begin{array}{ll}\cos A=\sin\,(90^\circ-A) & \sin A=\cos\,(90^\circ-A)\\ \cot A=\tan\,(90^\circ-A) & \tan A=\cot\,(90^\circ-A)\\ \csc A=\sec\,(90^\circ-A) & \sec A=\csc\,(90^\circ-A)\end{array}\right\}\cdot\quad[2]$$

RELATIONS OF THE SIX FUNCTIONS TO ONE ANOTHER.

10. From [1] we learn that the sine and the cosecant of an angle are reciprocals of each other; also that the same relation exists between the tangent and the cotangent, and between the secant and the cosine. Thus,

$$\left.\begin{array}{cc}\multicolumn{2}{c}{\sin A\csc A=1}\\ \csc A=\dfrac{1}{\sin A} & \sin A=\dfrac{1}{\csc A}\\ \multicolumn{2}{c}{\tan A\cot A=1}\\ \cot A=\dfrac{1}{\tan A} & \tan A=\dfrac{1}{\cot A}\\ \multicolumn{2}{c}{\sec A\cos A=1}\\ \cos A=\dfrac{1}{\sec A} & \sec A=\dfrac{1}{\cos A}\end{array}\right\}\cdot\cdot\cdot\quad[3]$$

11. From [1] we further learn that the tangent of an angle is the quotient of its sine divided by its cosine. Thus,

$$\sin A\div\cos A=\frac{a}{h}\div\frac{b}{h}=\frac{a}{h}\times\frac{h}{b}=\frac{a}{b}=\tan A.$$

---

[*] References to formulas are printed in brackets. The numbering of formulas is consecutive throughout the book.

We also learn that the cotangent is the quotient of the cosine divided by the sine. Thus,

$$\cos A \div \sin A = \frac{b}{h} \div \frac{a}{h} = \frac{b}{h} \times \frac{h}{a} = \frac{b}{a} = \cot A.$$

Hence

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A} \quad . \quad . \quad . \quad . \quad . \quad [4]$$

12. In the right-triangle ABC (Fig. 1) we know that

$$\overline{CB}^2 + \overline{AC}^2 = \overline{AB}^2,$$

or

$$a^2 + b^2 = h^2 \; . \; . \; . \; . \; . \; . \; . \; . \quad [5]$$

Dividing this equation through by h^2 gives

$$\frac{a^2}{h^2} + \frac{b^2}{h^2} = 1, \text{ or } \left(\frac{a}{h}\right)^2 + \left(\frac{b}{h}\right)^2 = 1.$$

But $\frac{a}{h} = \sin A$, and $\frac{b}{h} = \cos A$.

Therefore, by substitution,

$$(\sin A)^2 + (\cos A)^2 = 1$$

that is, the square of the sine of an angle plus the square of its cosine equals 1.

This formula is for brevity usually written without parentheses, thus:

$$\sin^2 A + \cos^2 A = 1. \; . \; . \; . \; . \; . \quad [6]$$

13. From [6] are derived

$$\sin A = \sqrt{1 - \cos^2 A} = \sqrt{(1 + \cos A)\,(1 - \cos A)} \; . \; . \quad [7]$$

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{(1 + \sin A)\,(1 - \sin A)} \; . \; . \quad [8]$$

14. The equation $h^2 = b^2 + a^2$ divided through by b^2 gives a result which by [1] is reduced to

$$\sec^2 A = 1 + \tan^2 A \quad . \quad . \quad . \quad . \quad . \quad . \quad [9]$$

that is, the square of the secant of an angle is equal to 1 plus the square of the tangent.

15. The same equation divided through by a^2 and reduced by [1] gives

$$\csc^2 A = 1 + \cot^2 A \quad . \quad . \quad . \quad . \quad . \quad . \quad [10]$$

that is, the square of the cosecant of an angle is equal to 1 plus the square of the cotangent.

16. From [9] are derived

$$\sec A = \sqrt{1 + \tan^2 A} \quad . \quad . \quad . \quad . \quad . \quad . \quad [11]$$

$$\tan A = \sqrt{\sec^2 A - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [12]$$

17. From [10] are derived

$$\csc A = \sqrt{1 + \cot^2 A} \quad . \quad . \quad . \quad . \quad . \quad . \quad [13]$$

$$\cot A = \sqrt{\csc^2 A - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [14]$$

18. The foregoing formulas [3] to [14] show that the six trigonometric functions of an angle are so related that when the value of any one function is given, the values of the other five functions can be found.

Thus, suppose given

$$\sin A = 0.3746.$$

Then, by [8], $\cos A = \sqrt{1 - (0.3746)^2} = 0.9272$

by [4], $\tan A = \dfrac{0.3746}{0.9272} = 0.4040$

$\cot A = \dfrac{0.9272}{0.3746} = 2.475$

$$\text{by [3], } \sec A = \frac{1}{0.9272} = 1.0785$$

$$\csc A = \frac{1}{0.3746} = 2.669.$$

These results, expressed to the nearest thousandth, are found in Table I (p. 9), where they appear as the functions of 22°.

Again, suppose given

$$\tan A = 0.8391.$$

Then,

$$\text{by [3], } \cot A = \frac{1}{0.8391} = 1.19175$$

$$\text{by [11], } \sec A = \sqrt{1 + (0.8391)^2} = 1.3054$$

$$\text{by [13], } \csc A = \sqrt{1 + (1.19175)^2} = 1.5557$$

$$\text{by [3], } \cos A = \frac{1}{1.3054} = 0.7660$$

$$\sin A = \frac{1}{1.5557} = 0.6428.$$

These results, expressed to the nearest thousandth, are found in Table I as the functions of 40°.

For further illustration, any single value in Table I (p. 9) may be taken as a given value, and from it may be computed the other five values found in the same line with it.

Thus we see how all the values in five columns of Table I can be computed if the values in any one column are given. How the values in a single column are computed in the first place will be explained further on. (See Appendix A.)

TABLE I.

Natural Values of the Trigonometric Functions.

Angle.	**Sin.**	**Csc.**	**Tan.**	**Cot.**	**Sec.**	**Cos.**	
0°	0.000	infinite.	0.000	infinite.	1.000	1.000	**90°**
1	0.017	57.30	0.017	57.29	1.000	1.000	**89**
2	0.035	28.65	0.035	28.64	1.001	0.999	**88**
3	0.052	19.11	0.052	19.08	1.001	0.999	**87**
4	0.070	14.34	0.070	14.30	1.002	0.998	**86**
5°	0.087	11.47	0.087	11.43	1.004	0.996	**85°**
6	0.105	9.567	0.105	9.514	1.006	0.995	**84**
7	0.122	8.206	0.123	8.144	1.008	0.993	**83**
8	0.139	7.185	0.141	7.115	1.010	0.990	**82**
9	0.156	6.392	0.158	6.314	1.012	0.988	**81**
10°	0.174	5.759	0.176	5.671	1.015	0.985	**80°**
11	0.191	5.241	0.194	5.145	1.019	0.982	**79**
12	0.208	4.810	0.213	4.705	1.022	0.978	**78**
13	0.225	4.445	0.231	4.331	1.026	0.974	**77**
14	0.242	4.134	0.249	4.011	1.031	0.970	**76**
15°	0.259	3.864	0.268	3.732	1.035	0.966	**75°**
16	0.276	3.628	0.287	3.487	1.040	0.961	**74**
17	0.292	3.420	0.306	3.271	1.046	0.956	**73**
18	0.309	3.236	0.325	3.078	1.051	0.951	**72**
19	0.326	3.072	0.344	2.904	1.058	0.946	**71**
20°	0.342	2.924	0.364	2.747	1.064	0.940	**70°**
21	0.358	2.790	0.384	2.605	1.071	0.934	**69**
22	0.375	2.669	0.404	2.475	1.079	0.927	**68**
23	0.391	2.559	0.424	2.356	1.086	0.921	**67**
24	0.407	2.459	0.445	2.246	1.095	0.914	**66**
25°	0.423	2.366	0.466	2.145	1.103	0.906	**65°**
26	0.438	2.281	0.488	2.050	1.113	0.899	**64**
27	0.454	2.203	0.510	1.963	1.122	0.891	**63**
28	0.469	2.130	0.532	1.881	1.133	0.883	**62**
29	0.485	2.063	0.554	1.804	1.143	0.875	**61**
30°	0.500	2.000	0.577	1.732	1.155	0.866	**60°**
31	0.515	1.942	0.601	1.664	1.167	0.857	**59**
32	0.530	1.887	0.625	1.600	1.179	0.848	**58**
33	0.545	1.836	0.649	1.540	1.192	0.839	**57**
34	0.559	1.788	0.675	1.483	1.206	0.829	**56**
35°	0.574	1.743	0.700	1.428	1.221	0.819	**55°**
36	0.588	1.701	0.727	1.376	1.236	0.809	**54**
37	0.602	1.662	0.754	1.327	1.252	0.799	**53**
38	0.616	1.624	0.781	1.280	1.269	0.788	**52**
39	0.629	1.589	0.810	1.235	1.287	0.777	**51**
40°	0.643	1.556	0.839	1.192	1.305	0.766	**50°**
41	0.656	1.524	0.869	1.150	1.325	0.755	**49**
42	0.669	1.494	0.900	1.111	1.346	0.743	**48**
43	0.682	1.466	0.933	1.072	1.367	0.731	**47**
44	0.695	1.440	0.966	1.036	1.390	0.719	**46**
45°	0.707	1.414	1.000	1.000	1.414	0.707	**45°**
	Cos.	**Sec.**	**Cot.**	**Tan.**	**Csc.**	**Sin.**	**Angle.**

TABLES OF TRIGONOMETRIC FUNCTIONS.

19. Table I shows the plan upon which all tables of trigonometric functions are arranged. Values of the angle, differing by any chosen interval (here one degree), are entered in the left-hand margin of the page in regular order from 0° to 45°, and the corresponding values of the functions are entered opposite them in the parallel columns, with the names of the functions at the top. Then the values of the angle from 45° to 90° are entered in the right-hand margin, in such a way that each value stands in the same line with its complement already entered in the left-hand margin; and the corresponding values of the functions are found in the parallel columns, with the names of the functions at the bottom. Such a table is said to return upon itself. This double arrangement is possible because, as we learn from [2], each function of an angle between 45° and 90° is identical in value with some function of its complement, and this complement is an angle between 0° and 45°. Thus, the sine of 70° is equal to the cosine of 20°; and so we find (Table I) the value 0.940 at once in the line with 20° *under* the heading "cos," and in the line with 70° *over* the footing "sin." Again, the tangent of 60° is equal to the cotangent of 30°; and so we find 1.732 at once in the line with 30° *under* the heading "cot" and in the line with 60° *over* the footing "tan." Each value entered in the table serves a double purpose: it is a function (named at the top) of the angle in the same line with it at the left of the page; and it is also a function (named at the bottom) of the angle in the same line with it at the right of the page.

20. Tables differ from one another in the greater or less number of decimal places to which the values of the functions are expressed, and in the greater or less intervals of angles for which these values are given. The number of decimal places varies in different tables from three to seven; and the interval of angle is in small tables one degree, but in larger tables it is one minute, ten seconds, or even one second.

The larger tables usually contain, not the values of the functions themselves, but the logarithms of those values. The latter are called *logarithmic functions* to distinguish them from the former, which are called *natural functions.*

For purposes of illustration we shall for the present use Table I, which contains the natural functions for every degree of the quadrant, expressed to three decimal places; but later we shall refer to any larger tables the reader may be supposed to have.*

SIMPLE INTERPOLATION.

21. When the exact value of the angle whose function is sought is found in the margin of the table, the function is read directly from the table; but when the value of the angle falls between two values in the margin, the value of its functions must be computed from values given in the table by a process called simple interpolation. This process is based on the assumption that *small variations in the value of the function are proportional to the corresponding variations in the value of the angle.* The amount of error in this assumption is of no practical consequence in any work save the most refined computations of astronomy and geodesy.

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* As a companion to this book, the authors have compiled the "Logarithmic and Trigonometric Tables." These are five-place tables, containing (1) the logarithms of numbers from 1 to 11009, (2) the logarithms of trigonometric functions for every minute of the quadrant, and (3) the natural values of trigonometric functions for every tenth minute of the quadrant. Beside these three principal tables there are various smaller tables. Directions for using the tables are given at the end. All five-place logarithms used in the present work are taken from the tables found in this companion, which will be referred to, hereafter, by the abbreviation *Log. Trig. Tab.*

22. The nature of simple interpolation will best be understood from examples.

(i) To find, from Table I, the sine of 32° 40′.

The sine of 32° is 0.530, and the sine of 33° is 0.545. As the sine increases with the increase of the angle, the sine of 32° 40′ will be found by *adding* to the sine of 32° *a part of the difference* between that and the sine of 33°. This difference is 15 thousandths, and the part of it to be added is, on the principle above assumed, the same part of 15 thousandths that 40′ is of one of degree; that is $\frac{40}{60}$ of 15 thousandths, or 10 thousandths.

Thus:

sin 32° . . . . = 0.530
+ correction (= $\frac{40}{60}$ of 15 thousandths) = 10
sin 32° 40′ . . . . = 0.540

(ii) To find the cosine of 37° 20′.

Reasoning as before.

cos 37° . . . . = 0.799
— correction (= $\frac{20}{60}$ of 11 thousandths) = 4
cos 37° 20′ . . . . = 0.795

The correction is subtracted in this case, because the cosine decreases with an increase of the angle.

In general, as an inspection of Table I shows, *when the* *angle increases, the sine, tangent, and secant increase, but the cosine, cotangent, and cosecant decrease.* This statement is the guide in applying corrections.

(iii) To find the tangent of 73° 15′.

tan 73° . . . . = 3.271
+ correction ($=\frac{15}{60}$ of 216 thousandths) = 54
tan 73° 15′ . . . . = 3.325

(iv) To find the cotangent of 23° 45′.

cot 23° . . . . = 2.356
— correction ($=\frac{45}{60}$ of 110 thousandths) = 82
cot 23° 45′ . . . . = 2.274

(v) To find the secant of 43° 50′.

sec 43° . . . . = 1.367
+ correction ($=\frac{50}{60}$ of 23 thousandths) = 19
sec 43° 50′ . . . . = 1.386

(vi) To find the cosecant of 53° 10′.

csc 53° . . . . = 1.252
— correction ($=\frac{10}{60}$ of 16 thousandths) = 3
csc 53° 10′ . . . . = 1.249

23. Simple interpolation is likewise used when the value of an angle is to be found from a value of the function falling between two values given in the table. The process will be understood from examples.

(i) Given sin $A = 0.447$ to find A.

Looking in the column of sines (Table I), we find that 0.447 falls between 0.438, which is the sine of 26°, and 0.454, which is the sine of 27°. The sine of A exceeds the sine of 26° by 9 thousandths, and the whole interval between the sine of 26° and the sine of 27° is 16 thousandths. Accordingly, the value of A exceeds 26° by the same part of a degree that 9 thousandths is of 16 thousandths; that is, A equals $26\frac{9}{16}$ degrees, or 26° 34′.

The work may be arranged thus:

$$\begin{array}{rr} \sin 26^\circ = 0.438 & \sin 26^\circ = 0.438 \\ \sin A \;\;= \underline{0.447} & \sin 27^\circ = \underline{0.454} \\ 9 & 16 \end{array}$$

$$A = 26\tfrac{9}{16} \text{ degrees} = 26^\circ\ 34'.$$

(ii) Given $\cos A = 0.625$, to find A.

$$\begin{array}{rr} \cos 51^\circ = 0.629 & \cos 51^\circ = 0.629 \\ \cos A \;\;= \underline{0.625} & \cos 52^\circ = \underline{0.616} \\ 4 & 13 \end{array}$$

$$A = 51\tfrac{4}{13} \text{ degrees} = 51^\circ\ 18'.$$

(iii) Given $\tan A = 0.047$ to find A.

$$\begin{array}{rr} \tan 2^\circ = 0.035 & \tan 2^\circ = 0.035 \\ \tan A \;\;= \underline{0.047} & \tan 3^\circ = \underline{0.052} \\ 12 & 17 \end{array}$$

$$A = 2\tfrac{12}{17} \text{ degrees} = 2^\circ\ 42'.$$

(iv) Given $\cot A = 0.251$ to find A.

$$\begin{array}{rr} \cot 75^\circ = 0.268 & \cot 75^\circ = 0.268 \\ \cot A \;\;= \underline{0.251} & \cot 76^\circ = \underline{0.249} \\ 17 & 19 \end{array}$$

$$A = 75\tfrac{17}{19} \text{ degrees} = 75^\circ\ 54'.$$

(v) Given $\sec A = 3.340$ to find A.

$$\begin{array}{rr} \sec 72^\circ = 3.236 & \sec 72^\circ = 3.236 \\ \sec A \;\;= \underline{3.340} & \sec 73^\circ = \underline{3.420} \\ 104 & 184 \end{array}$$

$$A = 72\tfrac{104}{184} \text{ degrees} = 72^\circ\ 34'.$$

(vi) Given $\csc A = 1.428$ to find A.

$$\begin{array}{rr} \csc 44^\circ = 1.440 & \csc 44^\circ = 1.440 \\ \csc A \;\;= \underline{1.428} & \csc 45^\circ = \underline{1.414} \\ 12 & 26 \end{array}$$

$$A = 44\tfrac{12}{26} \text{ degrees} = 44^\circ\ 28'.$$

For further illustration of this subject, see *Log. Trig. Tab.*, pp. 157–160.

24. Exercises.

1. Find the values of the six functions of the smaller acute angle of a right triangle whose sides measure 6, 8, and 10 inches respectively.

2. The sides of a right triangle being 5, 12 and 13 inches long respectively, what are the values of the six functions of the greater acute angle?

3. Draw a right triangle making one of its acute angles equal by measurement to any whole number of degrees, as 35°. Measure the sides of this triangle and divide the lengths by one another to find the values of the six functions of 35°. Compare the results with the values given in Table I.

4. Draw a right triangle making the base 10 inches and the perpendicular 5.32 inches long. What functions of the acute angles can be found from these two lengths? Do the results so found agree with values given in Table I? What then are the values of the acute angles of this triangle? Does measurement give the same values?

5. Repeat the last exercise, putting in place of 5.32 ten times any value given in the column of tangents, Table I.

6. Given $\sin A = \frac{2}{3}$, to find the values of the other functions of A.

7. Given $\cos A = \sqrt{\frac{2}{3}}$, to find the values of the other function of A.

8. Given $\tan A = 3$, to find the values of the other functions of A.

9. Given $\cot A = \frac{3}{4}$, to find the values of the other functions of A.

10. Given $\sec A = 3$, to find the values of the other functions of A.

11. Given $\csc A = 2$, to find the values of the other functions of A.

12. From $\cos 25° = 0.9063$, compute the values of the other functions of 25°.

13. From $\cot 57° = 0.6494$, compute the values of the other functions of 57°.

14. Draw a square and one of its diagonals. Using one of the right triangles so formed, prove that

$$\sin 45° = \cos 45° = \sqrt{\tfrac{1}{2}} \qquad \tan 45° = \cot 45° = 1$$
$$\sec 45° = \csc 45° = \sqrt{2}\,*$$

15. Draw an equilateral triangle and a perpendicular from one vertex to the opposite side. Using one of the right triangles so formed, prove that

$$\begin{array}{ll} \sin 30° = \cos 60° = \tfrac{1}{2} & \cos 30° = \sin 60° = \tfrac{1}{2}\sqrt{3} \\ \tan 30° = \cot 60° = \tfrac{1}{3}\sqrt{3} & \cot 30° = \tan 60° = \sqrt{3} \\ \sec 30° = \csc 60° = \tfrac{2}{3}\sqrt{3} & \csc 30° = \sec 60° = 2\,* \end{array}$$

From Table I find by interpolation:

16. sin 6° 35′
17. sin 73° 18′
18. cos 19° 20′
19. cos 47° 40′
20. tan 11° 15′
21. tan 67° 20′
22. cot 23° 30′
23. cot 52° 25′
24. sec 13° 35′
25. sec 66° 45′
26. csc 31° 20′
27. csc 79° 36′

From Table I find by interpolation the value of A in each of the following equations:

28. $\sin A = 0.145$.
29. $\sin A = 0.825$.
30. $\cos A = 0.948$.
31. $\cos A = 0.100$.
32. $\sin A = \tfrac{2}{3}$.
33. $\cos A = \tfrac{3}{5}$.
34. $\tan A = 0.315$.
35. $\tan A = 1\tfrac{7}{8}$.
36. $\cot A = 2.700$.
37. $\cot A = 1\tfrac{2}{3}$.
38. $\sin A = \sqrt{\tfrac{2}{5}}$.
39. $\cos A = \sqrt{\tfrac{2}{3}}$.
40. $\sec A = \tfrac{1}{2}\sqrt{5}$.
41. $\sec A = 1\tfrac{1}{5}$.
42. $\csc A = \sqrt{10}$.
43. $\csc A = 3\tfrac{3}{4}$.
44. $\sin A = \tfrac{1}{2}\sqrt{3}$.
45. $\cos A = \tfrac{1}{3}\sqrt{5}$.

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* Reduce these values to decimals and compare them with those in Table I.

CHAPTER II.

THE SOLUTION OF RIGHT TRIANGLES.

25. The right angle being a known part, it is only necessary that one side and one other part of a right triangle be given, that the remaining parts may be found. The given parts of a right triangle, then, may be:

I. The hypothenuse and an acute angle, as h and A.
II. One leg and the acute angle opposite, as a and A.
III. One leg and the acute angle adjacent, as b and A.
IV. The hypothenuse and one leg, as h and a.
V. The two legs, a and b.

26. The equations to be used for computing the required parts are the definitions of the functions [1] and the equation $a^2 + b^2 = h^2$. In selecting the particular equation to be used in any case, the rule should be to connect each required part separately with the given part. When one of the required parts becomes known through computation, it should not be used as a given part in a subsequent computation, lest a possible error in computation be perpetuated. There is only one case in which there is any reason for disregarding this precept.

I.

27. Given h and A, to find B, a, and b.

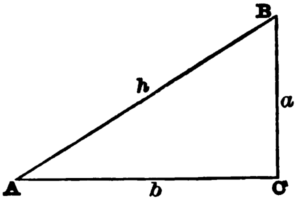

Fig. 2.

The other acute angle is found by subtracting the given acute angle from 90°; that is

$$B=90^\circ-A \quad . \quad . \quad . \quad . \quad . \quad . \qquad [15]$$

To find the leg a, select from [1] an equation connecting a with the given parts A and h. It is

$$\sin A=\frac{a}{h}$$

which gives

$$a=h \sin A \quad . \quad . \quad . \quad . \quad . \quad . \qquad [16]$$

To find b, select from [1] an equation connecting b and the given parts A and h. It is

$$\cos A=\frac{b}{h}$$

which gives

$$b=h \cos A \quad . \quad . \quad . \quad . \quad . \quad . \qquad [17]$$

28. EXAMPLE. Given $h=53$ and $A=23^\circ\ 40'$ to find the other parts.

By [15], $B=90^\circ-23^\circ\ 40'=66^\circ\ 20'$.
By [16], $a=53 \times 0.402 =21.31$
By [17], $b=53 \times 0.916 =48.55$

II.

29. Given a and A, to find B, h, and b.

The other acute angle is found by [15].

To find h and b, select from [1] the equations

$$\csc A=\frac{h}{a} \qquad\qquad \cot A=\frac{b}{a}$$

which give

$$h=a \csc A \quad . \quad . \quad . \quad . \quad . \quad . \qquad [18]$$

$$b=a \cot A \quad . \quad . \quad . \quad . \quad . \quad . \qquad [19]$$

30. EXAMPLE. Given $a=28$ and $A=49°\ 20'$, to find the other parts.

By [15], $B=90°-49°\ 20'=40°\ 40'$.
By [18], $h=28\times 1.318=36.90$.
By [19], $b=28\times 0.859=24.05$.

III.

31. Given b and A, to find B, h, and a.

The other acute angle is found by [15].

To find h and a, select from [1] the equations

$$\sec A=\frac{h}{b}\qquad \tan A=\frac{a}{b}$$

which give

$$h=b\sec A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [20]$$

$$a=b\tan A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [21]$$

32. EXAMPLE. Given $b=110$, and $A=76°\ 30'$, to find the other parts.

By [15], $B=90°-76°\ 30'=13°\ 30'$.
By [20], $h=110\times 4.289=471.8$.
By [21], $a=110\times 4.171=458.8$.

IV.

33. Given h and a, to find A, B, and b.

To find the angles, select from [1] the equations.

$$\sin A=\cos B=\frac{a}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad [22]$$

To find b use the equation

$$a^2+b^2=h^2$$

changing it to the more convenient form

$$b=\sqrt{(h+a)\ (h-a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad [23]$$

34. **Example.** Given $h=45$ and $a=37$, to find the other parts.

$$\text{By [22], } \sin A=\cos B=\frac{37}{45}=0.822.$$

$$A=55°\ 18'. \quad B=34°\ 42'.$$

$$\text{By [23], } b=\sqrt{(45+37)(45-37)}=25.61.$$

V.

35. Given a and b, to find A, B, and h.

To find the angles, select from [1] the equations

$$\tan A=\cot B=\frac{a}{b} \quad \ldots\ldots \quad [24]$$

The hypothenuse may be found by the equation

$$h=\sqrt{a^2+b^2} \quad \ldots\ldots \quad [25]$$

but when the given values of a and b are large numbers it is more convenient to find h by one of the equations.

$$h=a\csc A, \quad h=b\sec A \quad \ldots\ldots \quad [26]$$

which may be used after A has been computed. This is the case referred to on page 17, in which it is not always best to compute directly from the data.

36. **Example.** Given $a=50$ and $b=75$, to find the other parts.

$$\text{By [24], } \tan A=\cot B=\frac{50}{75}=0.667.$$

$$A=33°\ 42'. \quad B=56°\ 18'.$$

$$\text{By [25], } h=\sqrt{50^2+75^2}=90.14.$$

$$\text{By [26], } h=50\times 1.803^*=90.15.$$

$$h=75\times 1.202^*=90.15.$$

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[*] Instead of interpolating for 42′ in the last two computations, interpolate for $\frac{18}{26}$ of a degree, which is more exact.

37. Exercises.

1. Find the other parts of a right triangle of which the hypothenuse measures 16 feet, and one acute angle 47° 36′.

2. Find the other parts of a right triangle of which the hypothenuse measures 6 feet 8 inches, and an acute angle 10° 14′.

3. Find the other parts of a right triangle of which the perpendicular measures 35 feet, and the angle opposite thereto 22° 55′.

4. Find the other parts of a right triangle of which the base measures 27 feet 6 inches, and the angle opposite thereto 66° 42′.

5. Find the other parts of a right triangle of which the perpendicular measures 42.5 feet, and the acute angle adjacent thereto 35° 15′.

6. Find the other parts of a right triangle of which the base measures 3.46 metres, and the acute angle adjacent thereto 39° 45′.

7. Find the other parts of a right triangle of which the hypothenuse measures 32 feet and one leg 28 feet.

8. Find the other parts of a right triangle of which the hypothenuse measures 6.7 inches and one leg 4.8 inches.

9. Find the other parts of a right triangle of which one leg measures 85 feet and the other 204 feet.

10. Find the other parts of a right triangle of which one leg measures 0.362 metres and the other 0.436 metres.

11. A ladder 35 feet long leans against the side of a house, the foot of the ladder being 12 feet away from the house. How high does the ladder reach? What angle does it make with the side of the house?

12. A beam 28 feet long rests with one end on a level floor and the other raised 20 inches above it. What is the inclination of the beam to the floor?

13. Two stakes stand 20 feet apart, and the top of one is 3 feet higher than the top of the other. What is the length of a line from the top of one to the top of the other, and what angle does this line make with the horizontal?

14. A road which rises 1 foot in a horizontal distance of 20 feet is inclined to the level how many degrees?

15. A barn is to be 40 feet wide and the roof is to slope equally both ways from the middle. How high must the ridge-pole be above the level of the eaves that the slope of the roof may be 35°?

16. A room being 18 feet long, 15 feet wide, and $10\frac{1}{2}$ feet high, suppose a line drawn from one corner of the floor to the diagonally opposite corner of the ceiling. Find the angles which this line makes with each edge of the floor, and also with the diagonal of the floor drawn from the same corner with it.

⁂ A surveyor indicates the direction of lines by reference to the meridian, or north and south line, passing through the place of observation. If he is facing the north, lines running off to his right in any direction between due north and due east are described as "bearing" north so many degrees east, the number of degrees being the measure of the angle between the meridian and the observed line; and lines running off to his left in any direction between due north and due west are described as "bearing" north so many degrees west. In the same way if he is facing south, lines are described as bearing south so many degrees east or west. This enables him to describe lines of all directions, and the observed angle needs never exceed 90°.

17. A surveyor measures a line running north 42° east 120 rods. The end of this line is how far north, and how far east of the beginning?

18. A man walks in a direction north 48° west. How far north and how far west of his starting-point will he be when he has walked 200 rods?

19. The town of B is 40 miles due north of the town of A, and the town of C is 36 miles due east of B. What is the direction of C from A? What is the distance?

20. A man walks in a direction south 39° east until he is 75 rods further south than he was at the starting-point. How far is he to the east of his starting-point? How far has he walked?

21. A ship sails from a harbor on a course N. E. by N.* How far has she sailed when a certain headland, known to be 34 miles due north from the harbor, is seen bearing due west from the ship? How far away is the headland?

22. Two men start from the same point, one riding due south 7 miles an hour, and the other due east $8\frac{1}{2}$ miles an an hour. How far apart will they be at the end of 9 hours, and in what direction is the first from the second at any time during the 9 hours?

23. A surveyor runs a line south 32° 15′ east 14.3 rods, then from the end of this another line south 67° 50′ west 21.4 rods, and from the end of this last another line north 15° 20′ west 8.6 rods. From the point he has now reached compute the bearing and the distance of his starting-point.

24. A surveyor in order to find the breadth of a river drives a stake at a point directly opposite a tree on the other bank. At a convenient distance from this stake he drives another, in such a position that the line joining the two stakes makes a right angle with the line from the first stake to the tree. Then he measures the angle, 53° 34′, which the line from the second stake to the tree makes with the line joining the two stakes, and the distance, 43.6 rods, between the two stakes. Find the breadth of the river.

25. A train of cars was running on a straight railroad track across a prairie in a direction due west. At a point 5 miles north of the track a surveyor with his compass took the bearing of the engine, and 21 minutes 37 seconds later took it again. The first bearing was south 58° east and the second south 61° west. At what rate was the train running?

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* That is, north 33° 45′ east. The reader who is not familiar with the points of the mariner's compass will find a brief description in the Appendix.

⁂ The *angle of elevation* of an object is the observed angle which is formed by two lines proceeding from the point of observation, one inclining upward to the object observed, and the other running horizontally to a point directly under the object observed.

26. At a horizontal distance of 200 feet from the foot of a flag-staff the angle of elevation of the top is 34° 40′. How high is the flag-staff?

27. At a horizontal distance of 2 miles the top of a mountain is observed to be elevated at an angle of 7° 38′. How many feet high is the mountain?

28. At how great a horizontal distance from the foot of a tower 128 feet high will its top have an elevation of 28° 40′?

29. When the top of a monument 320 feet high is seen at an elevation of 7° 15′ by an observer standing on the same level with the base, how far off is the monument? How far must the observer walk towards it on the same level to change the angle of elevation to 20°?

⁂ The *angle of depression* of an object is the observed angle which is formed by two lines proceeding from the point of observation, one inclining downward to the object observed, and the other running horizontally to a point directly over the object observed.

30. From a church belfry known to be 250 feet above the level of the water in the harbor a boat is seen, the angle of depression being 3° 20′. How far away is the boat?

31. Two towers are 528 feet apart. From the top of one the angle of elevation of the top of the other is 12° 50′ and the angle of depression of the base 21° 35′. Find the height of each tower.

32. From the top of a bluff 236 feet above the sea level a yacht is seen sailing directly away; and in forty-five seconds she changes the angle of depression from 15° to 10°. How many miles an hour is she sailing?

⁂ It often happens that the point directly under the object whose angle of elevation is observed cannot be reached, so that its distance from the point of observation cannot be directly measured. Such a case would be that of a hill-top or mountain summit. Where it is practicable to select two stations on the same level, and in a line with the object to be observed, the angles of elevation taken at the two stations and the measured distance between the stations are sufficient data for computing the height and distance of the observed object. Let A denote the angle of elevation at the more distant station, B that at the nearer station, d the measured distance between the two stations, x the unknown distance from the nearer station to the point under the object observed, and y the unknown height of the object above the level of the stations. Then the two equations

$$\frac{y}{d+x}=\tan A \qquad \frac{y}{x}=\tan B$$

which are easily derived from the conditions stated, give, by the elimination of x,

$$y=\frac{d \tan B \tan A}{\tan B-\tan A}.$$

33. From two stations at the sea level the angles of elevation of the top of a lighthouse standing on a bluff and in a line with the two stations, are found to be 12° 44′ and 23° 19′; and the distance between the stations is 634 feet. Find the height of the top of the lighthouse above the sea level.

34. From a window on one side of the street the angle of depression is taken of the doorstep to the house directly opposite, and found to be 7° 10′. From another window directly over the first, and 19 feet higher up, the angle of depression of the same doorstep is found to be 15° 25′. How wide is the street? How high is each window above the street?

35. From a church belfry the angle of depression of the top and of the base of a monument are observed to be 6° 32′ and 13° 50′ respectively. How high is the belfry above the level of the base of the monument if the monument itself is 40 feet high?

36. From the deck of a steamer, which was moving due north twelve miles an hour, a lighthouse was seen bearing north 42° west, and one hour later the same lighthouse bore south 28° west. How many minutes after the beginning of the hour was it when the steamer was nearest the lighthouse, and how near was she?

37. How many degrees above the horizon is the sun, when an upright post 10 feet high casts a shadow $17\frac{1}{2}$ feet long on level ground?

38. The sun being on the equator on a certain day in March at noon, the shadows of upright objects in a certain place are then observed to be just two-thirds as long as the objects are high. What is the latitude of the place?

39. At noon on a day when the sun is 10° 14′ south of the equator, an upright pole 30 feet high casts how long a shadow on level ground if the place is in 42° 21′ north latitude?

Isosceles Triangles.

⁂ An isosceles triangle may be divided into two equal right triangles by drawing a perpendicular from the vertex to the base. This perpendicular bisects the angle at the vertex and the base.

40. Find the angles of an isosceles triangle of which the base measures 15 feet, and each of the other sides 23 feet.

41. The base of an isosceles triangle measures 27 feet and the height 37 feet. Find the angles.

42. The base of an isosceles triangle measures 213 feet and the angle at the vertex 29° 20′. What is the height? The length of each of the other sides?

43. Find the base and the altitude of an isosceles triangle of which each of the equal sides measures 36 feet, and the angle included by them is 48° 30′.

44. The altitude of a right cone is 36 inches and the diameter of the base 22 inches. Find the angle at the vertex.

45. The angle at the vertex of a right cone is 24° 32′, and the height of the cone is 20 inches. Find the diameter of the base and the slant height.

46. A right pyramid 50 feet square at the base is 80 feet high. Find the inclination of each face of the pyramid to the base, also the inclination of each edge to the base. Find the angles which one edge makes with each of the others.

Regular Polygons.

⁂ A regular polygon is divided into equal isosceles triangles by drawing to its vertices radii of the circumscribed circle. Each of these isosceles triangles is divided into two equal right triangles by drawing radii of the inscribed circle. The angles of these triangles are easily found when we know how many sides the regular polygon has. Then, with the functions of these angles, the sides of the polygon and the radii of the circumscribed and inscribed circles can be computed from one another.

47. Find the angles of the equal right triangles into which a regular polygon of six [eight, ten, twelve, fifteen, twenty] sides may be divided in the manner above described.

48. Find the length of each side of a regular polygon of five [eight, ten, twelve, fifteen, twenty] sides, if the diameter of the circumscribed circle is 12 inches.

49. Find the length of each side of a regular polygon of four [five, eight, ten, twelve, twenty] sides, if the diameter of the inscribed circle is 16 inches.

50. If a regular polygon of seven sides is inscribed in a circle of which the radius is ten inches, how long is each side of the polygon?

51. A regular polygon of nine sides is inscribed in a circle whose diameter is nine feet. Find the length of each side of the polygon.

52. *A* regular polygon of n sides being inscribed in a circle of given radius r, prove that, if s be the side of the polygon,

$$s = 2\,r \sin \frac{180^\circ}{n}.$$

53. A regular polygon of n sides being circumscribed about a circle of given radius r, prove that if s be the side of the polygon,

$$s = 2\,r \tan \frac{180^\circ}{n}.$$

54. Prove that if a regular polygon of n sides be inscribed in a circle, and another of the same number of sides be circumscribed about the same circle, the perimeter of the first polygon is to that of the second as the sine of $\frac{180^\circ}{n}$ is to the tangent of $\frac{180^\circ}{n}$.

55. If a regular polygon has ten sides, what is the ratio of the radius of the inscribed circle to the radius of the circumscribed circle? What, if the number of sides be four? five? six? eight? twelve? twenty?

56. Prove that the radius of the circle inscribed in a regular polygon of n sides is to the radius of the circle circumscribed about the same polygon as the cosine of $\frac{180^\circ}{n}$ is to 1.

57. Prove that the circumference of a circle is to the perimeter of a regular polygon inscribed in it as π is to $n \sin \frac{180^\circ}{n}$ where n is the number of sides of the polygon.

58. Prove that the circumference of a circle is to the perimeter of a regular polygon circumscribed about it as π is to $n \tan \frac{180^\circ}{n}$, where n is the number of sides of the polygon.

LOGARITHMIC SOLUTION OF RIGHT TRIANGLES.

38. The results obtained by the use of three-place functions, although regarded as only rough approximations, are yet as close as, or even closer than, the results obtained by geometrical drawing. Closer approximations require the values of the functions to be expressed to a greater number of decimal places, and to be tabulated for smaller intervals of angle. The increased arithmetical work which the use of such values would entail is avoided by the use of logarithms. The following examples will illustrate the methods of logarithmic solution.

I.

39. Given h and A, to find the other parts.

Since the logarithms of equals are equal, formulas [16] and [17], by taking the logarithm of each member, become

$$\log a = \log h + \log \sin A \quad . \quad . \quad . \quad . \quad [27]$$

$$\log b = \log h + \log \cos A \quad . \quad . \quad . \quad . \quad [28]$$

40. Example. Given $A = 38° 47' 35''$ and $h = 276.53$.

By [15], $B = 90° - 38° 47' 35'' = 51° 12' 25''$.

By [27]		By [28]	
log h	$=2.44174$	log h	$=2.44174$
log sin A	$=9.79693$	log cos A	$=9.89177$
log a	$=2.23867$*	log b	$=2.33351$*
a	$=173.25$	b	$=215.53$

---

* The logarithms of trigonometric functions are usually given in the tables with their characteristics increased by 10, particularly when the true characteristics are negative. This fact should be borne in mind when finding the characteristic of any sum of which such logarithms form a part, as in this example.

41. Test. The accuracy of these results may be tested by finding whether the computed values of a and b satisfy the equation $a^2 + b^2 = h^2$. But for convenience of logarithmic computation this equation should be changed to one of the forms

$$a^2 = (h + b)\ (h - b)$$
$$b^2 = (h + a)\ (h - a)$$

which, by taking the logarithm of each member, become

$$2 \log a = \log (h + b) + \log (h - b) \quad . \quad . \quad . \quad . \quad . \quad [29]$$
$$2 \log b = \log (h - a) + \log (h + a) \quad . \quad . \quad . \quad . \quad . \quad . \quad [30]$$

either of which may be used as a test. Thus:

By [29]		By [30]	
$h + b =$	492.06	$h + a =$	449.78
$h - b =$	61.00	$h - a =$	103.28
$\log (h + b) =$	2.69202	$\log (h + a) =$	2.65300
$\log (h - b) =$	1.78533	$\log (h - a) =$	2.01402
	4.47735		4.66702
$2 \log a =$	4.47734	$2 \log b =$	4.66702
Difference	1	Difference	0

II.

42. Given a and A to find the other parts.

By taking the logarithm of each member, [18] and [19] become

$$\log h = \log a + \log \csc A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [31]$$
$$\log b = \log a + \log \cot A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [32]$$

43. Example. Given $A = 12° \ 13' \ 47''$ and $a = 49.834$.

By [15], $B = 90° - 12° \ 13' \ 47'' = 77° \ 46' \ 13''$.

By [31]	By [32]
$\log a = 1.69753$	$\log a = 1.69753$
$\log \csc A = 0.67401$*	$\log \cot A = \underline{10.66404}$
$\log h = 2.37154$	$\log b = 2.36157$
$h = 235.256$	$b = 229.916$

44. TEST. [By 30]

$h + a = 285.090$	$\log (h + a) = 2.45498$
$h - a = 185.422$	$\log (h - a) = \underline{2.26816}$
	4.72314
	$2 \log b = 4.72314$
	Difference 0

III.

45. Given b and A, to find the other parts.

By taking the logarithm of each member, [20] and [21] become

$$\log h = \log b + \log \sec A \quad . \quad . \quad . \quad . \quad [33]$$

$$\log a = \log b + \log \tan A \quad . \quad . \quad . \quad . \quad [34]$$

---

* The logarithms of secants and of cosecants are not usually given in the tables; but they are easily found from the logarithms of the cosines and of the sines. Since the cosecant is the reciprocal of the sine, the logarithm of the cosecant is the negative of the logarithm of the sine; that is,

$$\log \csc A = -\log \sin A.$$

What we find in the tables is always the logarithm of the sine with its characteristic increased by 10, that is $10 + \log \sin A$, which may be called *the tabular log sin A*. If this *tabular log sin A* be subtracted from 10 the result, $10 - (10 + \log \sin A)$, is equal to $-\log \sin A$, which by the equation above given is the same as $\log \csc A$. Hence

$$\log \csc A = 10 - \text{tabular} \log \sin A.$$

In the same way,

$$\log \sec A = 10 - \text{tabular} \log \cos A.$$

46. EXAMPLE. Given $b=438.62$, and $A=57° \ 44' \ 35''$.

$$B=90°-57° \ 44' \ 35''=32° \ 15' \ 25''.$$

By [33]		By [34]	
log b	$=2.64209$	log b	$=2.64209$
log sec A	$=0.27269$*	log tan A	$=10.19988$
log h	$=2.91478$	log a	$=2.84197$
h	$=821.82$	a	$=694.98$

47. TEST. By [30]

$h+b$	$=1260.44$	log $(h+b)$	$=3.10052$
$h-b$	$=383.20$	log $(h-b)$	$=2.58343$
			5.68395
		2 log a	$=5.68394$
		Difference	1

IV.

48. Given h and a, to find the other parts.

By taking the logarithm of each member, [22] and [23] become

$$\log \sin A=\log a-\log h$$

$$\text{tabular} \log \sin A=\log a+\text{colog}\ h \quad . \ . \ . \ . \ . \tag{35}$$

$$\log b=\tfrac{1}{2}\left[\log (h+a)+\log (h-a)\right] \quad . \ . \tag{36}$$

49. EXAMPLE. Given $h=12436$, and $a=7632$.

By [35]		By [36]	
log a	$=3.88264$	$h+a$	$=20068$
colog h	$=5.90532$	$h-a$	$=4804$
log sin A	$=9.78796$	log $(h+a)$	$=4.30251$
A	$=37° \ 51' \ 28''$.	log $(h-a)$	$=3.68160$
B	$=52° \ 8' \ 32''$.		$2)7.98411$
		log b	$=3.99206$
		b	$=9818.7$

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* See note page 31.

50. **Test.** By the equation $a=b\tan A$, derived from [1], we shall have, if the work is right,

$$\log a=\log b+\log\tan A$$

$$\begin{array}{lr}
\log b & =\ 3.99206 \\
\log\tan A= & \underline{9.89059} \\
 & 3.88265 \\
\log a & =\ 3.88264 \\
\text{Difference} & 1
\end{array}$$

V.

51. Given a and b, to find the other parts.

By taking the logarithm of each member, [24] and [26] become

$$\begin{array}{rll}
\log\tan A= & \log a-\log b & \\
\text{tabular}\log\tan A= & \log a+\text{colog}\, b & \quad . \ . \ . \quad [37] \\
\log h\quad = & \log a+\log\csc A & \quad . \ . \quad [38]
\end{array}$$

52. **Example.** Given $a = 39.428$ and $b = 28.745$.

By [37]

$$\begin{array}{lr}
\log a & =\ 1.59581 \\
\text{colog}\, b & =\ \underline{8.54143} \\
\log\tan A & =10.13724 \\
A & =53^\circ\ 54'\ 21''. \\
B & =36^\circ\ \ 5'\ 39''.
\end{array}$$

By [38]

$$\begin{array}{lr}
\log a & =1.59581 \\
\log\csc A & =\underline{0.09256} \\
\log h & =1.68837 \\
h & =\ 48.794
\end{array}$$

53. **Test.** By [29]

$h+b=77.539$

$h-b=20.049$

$$\begin{array}{lr}
\log\ (h+b) & =\ 1.88952 \\
\log\ (h-b) & =\ \underline{1.30209} \\
 & 3.19161 \\
2\log a & =3.19162 \\
\text{Difference} & 1
\end{array}$$

54. Exercises.

1. Find the other parts of a right triangle of which the hypothenuse measures 12536 feet and one angle 4° 17′ 36″.

2. Find the other parts of a right triangle of which the base measures 33621 feet and the acute angle adjacent thereto 2° 57′ 35″.

3. Find the other parts of a right triangle of which the base is 12623 and the perpendicular 245 feet in length.

4. What is the inclination of a road which rises thirty-five feet in a mile?

5. Find the angles of a right triangle of which the hypothenuse is 76.375 meters and the base 76.106 meters.

6. Find the chord of an arc of 10°, if the diameter of the circle is 5362.7 feet.

7. How many feet high is a mountain which at a distance of 3 miles is elevated at an angle of 3° 10′ 15″? What would be its angle of elevation at a distance of two miles?

8. What is the distance from the centre of a circle to the middle of the chord which subtends an arc of 7° 43′ 18″, if the radius of the circle measures 876.43 meters? What is the length of the chord?

9. The height of an isosceles triangle being 325.6 feet, and the angle at the vertex 18° 19′ 44″, what is the length of the base?

10. Find one side of a regular polygon of 81 sides which is inscribed in a circle 3625.27 metres in diameter.

11. The equatorial radius of the earth being 20921665 feet, what is the length of the chord of one degree on the equator in miles? What is the distance between the middle of this chord and the surface of the earth?

12. A regular polygon of seven sides is circumscribed about a circle 628.52 meters in diameter. How long is each side of the polygon? What is the distance from the centre of the circle to one of the vertices of the polygon?

CHAPTER III.

THE SOLUTION OF OBLIQUE TRIANGLES BY DRAWING A PERPENDICULAR.

(55) (55)

55. An oblique triangle can be solved by drawing a perpendicular from one of its vertices to the opposite side or to the opposite side prolonged and solving the two right triangles so formed.

(55) The given parts of the oblique triangle may be

I. One side and two angles.
II. Two sides and the angle included by them.
III. Two sides and the angle opposite one of them.
IV. The three sides.

56. For uniformity in notation, the numerical values of the angles are denoted by the large letters A, B, and C; those of the sides respectively opposite them by the small letters a, b, and c; and the perpendiculars from the points **A, B,** and **C** upon the opposite sides are marked respectively **A P, B Q,** and **C R.**

I.

57. *Given one side and two angles of a triangle to find the other parts.*

Let b, A, and B be the given parts.

The third angle is found by subtracting the sum of the two given angles from 180°.

$$C = 180° - (A + B).$$

To find the side c, draw a perpendicular **C R** to it from the opposite vertex. This perpendicular falls within the triangle when both given angles are acute, Fig. 3, and without the triangle when either of the given angles is obtuse, Fig. 4.

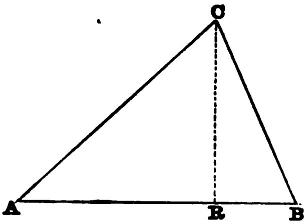

Fig. 3.

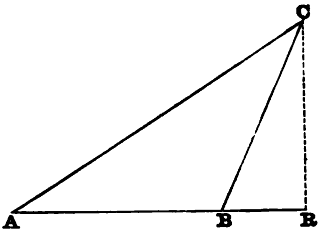

Fig. 4.

Then from the right triangles thus formed, we have:

In Fig. 3.	In Fig. 4.
$\mathrm{CR}=b\sin A$	$\mathrm{CR}=b\sin A$
$\mathrm{AR}=b\cos A$	$\mathrm{AR}=b\cos A$
$\mathrm{RB}=\mathrm{CR}\cot B$	$\mathrm{BR}=\mathrm{CR}\cot(180^\circ-B)$
$c=\mathrm{AR}+\mathrm{RB}$	$c=\mathrm{AR}-\mathrm{BR}$
$a=\mathrm{CR}\csc B$	$a=\mathrm{CR}\csc(180^\circ-B)$

58. Example. Given $A=40^\circ\ 20'$, $B=57^\circ\ 30'$, and $b=76$, to find the other parts.

$$C=180^\circ-(40^\circ\ 20'+57^\circ\ 30')=82^\circ\ 10'.$$

$$\begin{aligned}
\mathrm{CR}&=76\times 0.647^{*}=49.17\\
\mathrm{AR}&=76\times 0.762=57.91\\
\mathrm{RB}&=49.17\times 0.637=31.32\\
c&=57.91+31.32=89.23\\
a&=49.17\times 1.186=58.31
\end{aligned}$$

59. A figure may be drawn with scale and protractor for the purpose of showing the general correctness of the computation. Such a geometrical solution is a good safeguard against mistakes in the use of formulas, although

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* Natural functions are here used, taken from Table I., p. 9.

the measurements of the drawing can hardly test the results of computation beyond two figures.

60. The triangle may also be solved by drawing the perpendicular from the other end of the given side, that is, from A. The formulas to be used are easily found.* When applied to the above example they give the following results:

$$\begin{aligned}\mathrm{AP} &= 76 \times 0.990 = 75.24\\ \mathrm{CP} &= 76 \times 0.136 = 10.34\\ \mathrm{PB} &= 75.24 \times 0.637 = 47.93\\ a &= 10.34 + 47.93 = 58.27\\ c &= 75.24 \times 1.186 = 89.23\end{aligned}$$

61. The case in which the perpendicular falls without the triangle is illustrated by the following

EXAMPLE. Given $A = 29° \, 40'$, $B = 123° \, 10'$, and $b = 34$.

$$C = 180° - (29° \, 40' + 123° \, 10') = 27° \, 10'.$$

$$\begin{aligned}\mathrm{CR} &= 34 \times 0.495 = 16.83\\ \mathrm{AR} &= 34 \times 0.869 = 29.55\\ \mathrm{BR} &= 16.83 \times 0.653 = 10.99\\ c &= 29.55 - 10.99 = 18.56\\ a &= 16.83 \times 1.194 = 20.095\end{aligned}$$

62. If the same triangle be solved by drawing a perpendicular from A,* the results are as follows:

$$\begin{aligned}\mathrm{AP} &= 34 \times 0.456 = 15.50\\ \mathrm{CP} &= 34 \times 0.890 = 30.26\\ \mathrm{PB} &= 15.50 \times 0.653 = 10.12\\ a &= 30.26 - 10.12 = 20.14\\ c &= 15.50 \times 1.194 = 18.53\end{aligned}$$

These results agree nearly enough with the others to show that both sets are correct.

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* As an exercise, let the figure be drawn and the formulas found.

63. Exercises.

1. Find the other parts of a triangle, one side measuring 85 rods, and the angles adjacent 48° and 63°.

2. One side of a triangle is 93 feet long, the opposite angle measures 129° 38′, and one of the adjacent angles 19° 24′. Find the other sides.

3. Find the two sides of a triangle, if the angles opposite them measure 53° 14′ and 71° 44′, and the third side 27 rods.

4. The longest side of a triangle measures 17½ feet, the largest angle 78° 48′, and the smallest 47° 22′. Find the other parts.

5. The largest angle of a triangle measures 72°, and the other two differ by 10°; the shortest side measures 18 feet 9 inches. Find the other sides.

6. The base of a triangle is 4.68 inches long and the angles at the base measure 67° 25′ and 73° 48′. Find the two sides.

7. From the deck of a steamer a lighthouse is seen bearing N.N.W., and, after the steamer has gone ten miles on a course N.E. by E., the same lighthouse is seen bearing W.N.W. How far away was the lighthouse at first?

8. A line 100 feet long is measured along one bank of a river and at each end the angle is measured between that line and a line directed to a tree on the other bank. The angles measure 57° 28′ and 81° 39′. What is the distance from each end of the line to the tree?

9. A surveyor running a line due north comes to a large pond and must find the distance across. He sends a man in a boat who sets up a stake at the point where the line strikes the opposite shore. He next runs a line from his present station north 76° east 150 rods, and from the end of this line finds that the stake bears north 43° 36′ west. What is the distance across the pond?

II.

64. *Given two sides and the included angle of a triangle, to find the other parts.*

Let A, b, and c be the given parts.

Draw a perpendicular to the longer given side from the opposite vertex. Supposing c to be the longer given side, the perpendicular CR will fall within the triangle, Fig. 5, when A is acute, and without the triangle, Fig. 6, when A is obtuse.

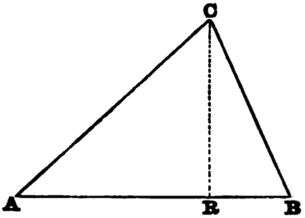

Fig. 5.

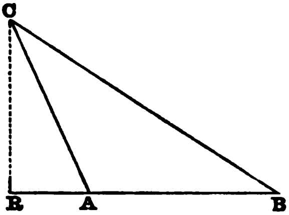

Fig. 6.

Then from the right triangles so formed we have

In Fig. 5,	In Fig. 6,
$\mathrm{CR} = b \sin A$	$\mathrm{CR} = b \sin (180^\circ - A)$
$\mathrm{AR} = b \cos A$	$\mathrm{AR} = b \cos (180^\circ - A)$
$\mathrm{RB} = c - \mathrm{AR}$	$\mathrm{BR} = c + \mathrm{AR}$

In both figures

$$\tan B = \frac{\mathrm{CR}}{\mathrm{RB}}$$

$$C = 180^\circ - (A + B)$$

$$a = \mathrm{CR} \csc B = \mathrm{RB} \sec B.$$

65. EXAMPLE. Given $A = 31^\circ\ 20'$, $b = 34$, and $c = 42$.

$$\mathrm{CR} = 34 \times 0.520 = 17.68$$

$$\mathrm{AR} = 34 \times 0.854 = 29.04$$

$$\mathrm{RB} = 42 - 29.04 = 12.96$$

$$\tan B = \frac{17.68}{12.96} = 1.364$$

$$B = 53° \ 45'$$

$$C = 180° - (31° \ 20' + 53° \ 45') = 94° \ 55'$$

$$a = 17.68 \times 1.240 = 21.92$$

$$= 12.96 \times 1.691 = 21.92$$

66. The triangle may also be solved by drawing a perpendicular BQ to the shorter given side. This perpendicular falls without the triangle when *A* is an obtuse angle; but when *A* is acute it may fall within or without. This point is settled by computing the length of AQ, and observing whether AQ is shorter or longer than the given side *b*. The formulas to be used are easily found.* When applied to the above example they give:

$$\text{BQ} = 42 \times 0.520 = 21.84$$

$$\text{AQ} = 42 \times 0.854 = 35.87,$$

and the perpendicular therefore falls without the triangle.

$$\text{CQ} = 35.87 - 34 = 1.87$$

$$\tan(180° - C) = \frac{21.84}{1.87} = 11.68$$

$$180° - C = 85° \ 5'$$

$$C = 94° \ 55'$$

$$B = 180° - (31° \ 20' + 94° \ 55') = 53° \ 45'$$

$$a = 21.84 \times 1.004 = 21.93$$

$$= 1.87 \times 11.71 = 21.90$$

67. The case in which the given angle is obtuse : illustrated by the following

EXAMPLE. Given $A = 143° \ 17'$, $b = 39$, and $c = 51$.

$$\text{CR} = 39 \times 0.598 = 23.32$$

$$\text{AR} = 39 \times 0.802 = 31.28$$

$$\text{BR} = 51 + 31.28 = 82.28$$

---

* As an exercise, let the figure be drawn and the formulas found.

$$\tan B = \frac{23.32}{82.28} = 0.283$$

$$B = 15^\circ\ 47'$$

$$C = 180^\circ - (143^\circ\ 17' + 15^\circ\ 47') = 20^\circ\ 56'$$

$$a = 23.32 \times 3.679 = 85.79$$

$$= 82.28 \times 1.039 = 85.49$$

68. By drawing a perpendicular BQ,* the following results are found:

$$BQ = 51 \times 0.598 = 30.50$$

$$AQ = 51 \times 0.802 = 40.90$$

$$CQ = 39 + 40.90 = 79.90$$

$$\tan C = \frac{30.50}{79.90} = 0.382$$

$$C = 20^\circ\ 54'$$

$$B = 180^\circ - (143^\circ\ 17' + 20^\circ\ 54') = 15^\circ\ 49'$$

$$a = 30.50 \times 2.803 = 85.49$$

$$= 79.90 \times 1.070 = 85.49$$

69. Exercises.

1. Two sides of a triangle measure 25 feet and 29 feet, and the included angle 64° 17′. Find the other parts.

2. Two sides of a triangle measure 37 rods and 41 rods. and the included angle 123° 42′. Find the other parts.

3. Sides 35 and 45, included angle 72° 40′.

4. Sides 112 and 109, included angle 118° 25′.

5. Sides 73.5 and 92.4, included angle 24° 18′.

6. Sides 63 and 22, included angle 158° 16′.

7. Sides 23.2 and 27.3, included angle 69° 39′.

8. Sides 6.5 and 5.6, included angle 98° 48′.

9. Sides 125 and 150, included angle 160°.

10. Sides 12.6 and 9.8, included angle 55° 55′.

11. Sides 19 and 20, included angle 32° 36′.

12. Sides 100 and 101, included angle 157° 15′.

13. From A to B is $11\frac{1}{2}$ miles, from A to C is $10\frac{1}{4}$ miles, and the angle B A C is 109° 35′. How far is it from B to C?

---

* As an exercise, let the figure be drawn and the formulas found.

14. A surveyor runs two lines from the same point, one north 37° 15′ east 104 rods, and the other north 28° 30′ west 136 rods. How far apart are the ends of these lines?

15. Two lines are run from the same point, one north 47° 34′ west 63 rods, and the other south 24° 10′ west 72 rods. How far apart are the ends of those lines?

16. From the same harbor two steamers start at the same time, one on a course N. by E. eleven miles an hour, and the other on a N. W. course nine miles an hour. If they hold these courses four hours, how far apart will they be at the end of that time?

17. A ship sails on a S. S. E. course 8 miles, and then on a N. E. course 16 miles. How far is she now from her starting-point?

18. The hour-hand of a clock is $10\frac{1}{2}$, and the minute-hand $11\frac{1}{2}$ inches long. How many inches apart are the ends of these hands at a quarter before one o'clock?

19. Three sides of a four-sided field can be measured, but the fourth side which runs through a swamp cannot. The following measurements are made: Side A B = 114 rods. Side B C = 132 rods. Side C D = 122 rods. Angle B = 81° 30′. Angle C = 74° 20′. From these data compute (1) the diagonal A C and the angle A C B; (2) the diagonal B D and the angle D B C; (3) the side A D, using the triangle A C D; (4) the side A D, using the triangle A B D.

20. Compute the fourth side of a field of which three sides measure 27, 32, and 36 rods, the angle included between the first two being 107° 15′, and that between the second and third 102° 25′.

21. From a point near the middle of a triangular field lines are run to the three corners, the first north 63° east 87.3 rods, the second south 8° east 78.4 rods, and the third north 76° west 119.7 rods. Compute the sides and angles of the field.

III.

70. *Given two sides and the angle opposite one of them to find the other parts of the triangle.*

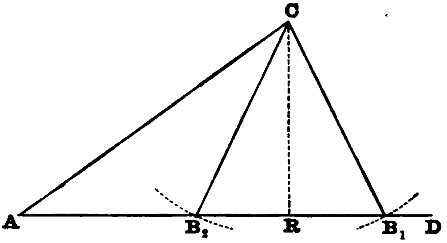

Fig. 7.

Let a, b, and A be the given parts. Draw a line **AD**, Fig. 7, of indefinite length, and at **A** construct an angle equal to the given angle and making the side **AC** equal to the given side b. With **C** as a centre and with a radius equal to the given side a, draw an arc. The two points B_1 and B_2 where this arc cuts the line **AD** determine two triangles AB_1C and AB_2C, each of which contains all the given parts. The problem, therefore, is said to have two solutions.

The equations for solution are derived from the right triangles formed by drawing **CR** perpendicular to **AD**. When the given angle A is acute, as represented in Fig. 7, these equations are:

$$
\begin{aligned}
\mathrm{CR} &= b \sin A \\
\mathrm{AR} &= b \cos A \\
\mathrm{RB} &= \sqrt{a^2 - \mathrm{CR}^2} = \sqrt{(a + \mathrm{CR})(a - \mathrm{CR})} \\
c &= \mathrm{AR} \pm \mathrm{RB}
\end{aligned}
$$

Since the square root of a quantity has two values numerically equal, but one positive and the other negative, **RB** must be taken with the double sign. The two values of **RB** are represented in Fig. 7 by RB_1 and RB_2

the opposite directions in which these lines extend from the point **R** corresponding to the opposite algebraic signs of the two values. When the positive value of **RB** is added to **AR** the resulting value of c is $\mathbf{AB_1}$, and when the negative value is added, the resulting value of c is $\mathbf{AB_2}$. Thus,

$$c_1 = \mathrm{AR} + \mathrm{RB} = \mathrm{AB_1}$$
$$c_2 = \mathrm{AR} - \mathrm{RB} = \mathrm{AB_2}$$

In the same way, the angle C has two values, $\mathbf{ACB_1}$ and $\mathbf{ACB_2}$. The first of these is the sum of **ACR** and $\mathbf{RCB_1}$, and the second is the difference between **ACR** and $\mathbf{RCB_2}$. The two angles $\mathbf{RCB_1}$, and $\mathbf{RCB_2}$ being equal and lying on opposite sides of the line **CR** may be considered to be positive and negative values of the same angle **RCB**; so that the two values of C may be expressed thus:

$$C_1 = \mathrm{ACR} + \mathrm{RCB} = \mathrm{ACB_1}$$
$$C_2 = \mathrm{ACR} - \mathrm{RCB} = \mathrm{ACB_2}$$

The part **ACR** is known from its being the complement of A.

$$\mathrm{ACR} = 90° - A.$$

The part **RCB** may be found from its sine or its cosine.

$$\sin \mathrm{RCB} = \frac{\mathrm{RB}}{a}, \qquad \cos \mathrm{RCB} = \frac{\mathrm{CR}}{a}$$

The angle B has two values, which, as the figure shows, are supplements of each other.

$$B_1 = 180° - (A + C_1)$$
$$B_2 = 180° - (A + C_2)$$

71. EXAMPLE. Given $a = 37$, $b = 47$, and $A = 49° \, 45'$.

$$\mathrm{CR} = 47 \times 0.763 = 35.86$$
$$\mathrm{AR} = 47 \times 0.646 = 30.36$$
$$a + \mathrm{CR} = 72.86$$

$$a - \mathrm{CR} = 1.14$$

$$\mathrm{RB} = \sqrt{72.86 \times 1.14} = \pm\, 9.11$$

$$c_1 = 30.36 + 9.11 = 39.47$$

$$c_2 = 30.36 - 9.11 = 21.25$$

$$\text{Angle } \mathrm{ACR} = 90^\circ - 49^\circ\, 45' = 40^\circ\, 15'$$

$$\sin \mathrm{RCB} = \frac{9.11}{37} = 0.246$$

$$\mathrm{RCB} = 14^\circ\, 14'$$

$$C_1 = 40^\circ\, 15' + 14^\circ\, 14' = 54^\circ\, 29'$$

$$C_2 = 40^\circ\, 15' - 14^\circ\, 14' = 26^\circ\ \ 1'$$

$$B_1 = 180^\circ - (49^\circ\, 45' + 54^\circ\, 29') = 75^\circ\, 46'$$

$$B_2 = 180^\circ - (49^\circ\, 45' + 26^\circ\ \ 1') = 104^\circ\, 14'$$

72. Special Cases.

(1) Referring to Fig. 8, we see that if the given side *a*, which is used as the radius in drawing the arc, were shorter than the perpendicular **CR**, there would be no triangles; because the arc would not cut the line **AD** at all, and the points B_1 and B_2 could not be found.

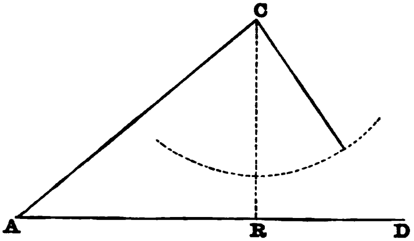

Fig. 8.

To show how the formulas indicate this, suppose that $a = 12$, $b = 35$, and $A = 32^\circ$. Then,

$$\mathrm{CR} = 35 \times 0.530 = 18.55$$

$$\mathrm{RB} = \sqrt{(12)^2 - (18.55)^2}$$

Thus **RB** is the square root of a negative number. The arithmetical impossibility of finding the square root of a negative number answers to the geometric impossibility of cutting a given straight line by an arc drawn from a given point as centre with a radius shorter than the shortest distance from that point to the line.

Regarding the general problem as having two solutions, we say that both solutions are impossible when the given angle A is acute, and the given side a is less than the perpendicular **CR**, that is, less than $b \sin A$.

Briefly stated, the conditions of this special case are

$$A < 90^\circ \text{ and } a < b \sin A^*.$$

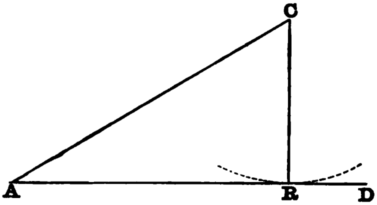

Fig. 9.

(2) If the given side a were equal to the perpendicular **CR**, Fig. 9, there would be only one triangle. For, in that case, the arc would touch the base **AD** at **R**, the two points $\mathbf{B_1}$ and $\mathbf{B_2}$ would fall together at **R** and the two triangles $\mathbf{AB_1C}$ and $\mathbf{AB_2C}$ would be one and the same right triangle **ARC**.

To show how the formulas indicate this, suppose $a = 19.4$, $b = 20$, and $A = 76^\circ$.

Then,

$$\begin{aligned}
\text{CR} &= 20 \times 0.970 = 19.4 \\
\text{AR} &= 20 \times 0.242 = 4.84 \\
\text{RB} &= \sqrt{(19.4)^2 - (19.4)^2} = \pm 0 \\
c &= 4.84 \pm 0 \\
\text{ACR} &= 90^\circ - 76^\circ = 14^\circ
\end{aligned}$$

---

* The sign < is read "is less than." Turned the other way, >, it is read "is greater than." Thus $A < B$ means that A is less than B, and $A > B$ means that A is greater than B.

$$\sin \mathrm{RCB} = \frac{0}{19.4} = 0$$
$$\mathrm{RCB} = 0°.$$
$$C = 14° \pm 0°$$
$$B = 180° - (76° + 14°) = 90°.$$

The conditions of this special case are

$$A < 90° \text{ and } a = b \sin A.$$

(3) If the given side a is greater than the perpendicular **CR**, but less than the given side b, as in the example first given, Fig. 7, there are two triangles each containing all the given parts. The conditions of this special case are

$$A < 90° \text{ and } b > a > b \sin A$$

(4) If the given side a were equal to the given side b, the first triangle **AB$_1$C**, Fig. 10, would be isosceles; while the second triangle **AB$_2$C** would be only a straight line, the point **B$_2$** falling at **A**.

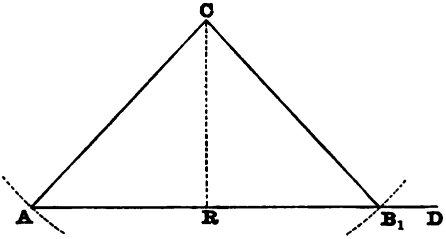

Fig. 10.

To show how the formulas indicate this, suppose $a = 15$, $b = 15$, and $A = 38°$. Then,

$$\mathrm{CR} = 15 \times 0.616 = 9.24$$
$$\mathrm{AR} = 15 \times 0.788 = 11.82$$
$$\mathrm{RB} = \sqrt{(15)^2 - (9.24)^2} = 11.82$$
$$c_1 = 11.82 + 11.82 = 23.64$$
$$c_1 = 11.82 - 11.82 = 0$$
$$\mathrm{ACR} = 90° - 38° = 52°$$
$$\sin \mathrm{RCB} = \frac{11.82}{15} = 0.788$$

$$
\begin{aligned}
\text{RCB} &= 52^\circ \\
C_1 &= 52^\circ + 52^\circ = 104^\circ \\
C_2 &= 52^\circ - 52^\circ = 0^\circ \\
B_1 &= 180^\circ - (38^\circ + 104^\circ) = 38^\circ \\
B_2 &= 180^\circ - (38^\circ + 0^\circ) = 142^\circ
\end{aligned}
$$

The conditions of this special case are

$$A < 90^\circ \text{ and } a = b.$$

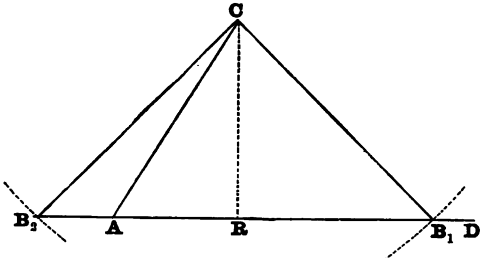

Fig. 11.

(5) If the given side a were longer than the given side b, the points **B$_1$** and **B$_2$** would fall in opposite directions from the point **A**, Fig. 11, the first in the base **AD**, the second in its prolongation. The two triangles **A·B$_1$ C** and **AB$_2$C** are both *possible* triangles; but one of them **AB$_2$C** is *inadmissible*, because it does not contain the given angle A as one of its parts.

To illustrate, suppose $a = 14.8$, $b = 13.5$ and $A = 54^\circ$.

Then, Fig. 11,

$$
\begin{aligned}
\text{CR} &= 13.5 \times 0.809 = 10.92 \\
\text{AR} &= 13.5 \times 0.588 = 7.94 \\
\text{RB} &= \sqrt{(14.8)^2 - (10.92)^2} = 9.99 \\
c_1 &= 7.94 + 9.99 = 17.93 \\
[c_2 &= 7.94 - 9.99 = -2.05] \\
\text{ACR} &= 90^\circ - 54^\circ = 36^\circ \\
\sin \text{RCB} &= \frac{9.99}{14.8} = 0.675
\end{aligned}
$$

$$\begin{aligned}
\text{RCB} &= 42^\circ\ 28' \\
C_1 &= 36^\circ + 42^\circ\ 28' = 78^\circ\ 28' \\
[C_2 &= 36^\circ - 42^\circ\ 28' = -6^\circ\ 28'] \\
B_1 &= 180^\circ - (54^\circ + 78^\circ\ 28') = 47^\circ\ 32' \\
[B_2 &= 180^\circ - (54^\circ - 6^\circ\ 28') = 132^\circ\ 28']
\end{aligned}$$

The negative values here obtained are represented in Fig. 11.

The conditions of this special case are

$$A < 90^\circ \text{ and } a > b.$$

73. Thus far the angle A has been assumed to be acute. If it is obtuse, the perpendicular falls without the triangle, Fig. 12, on the prolongation of the base, so that in the right triangle **ARC** we have, not the given angle A, but its supplement $180^\circ - A$. The formulas for solution derived from this figure are:

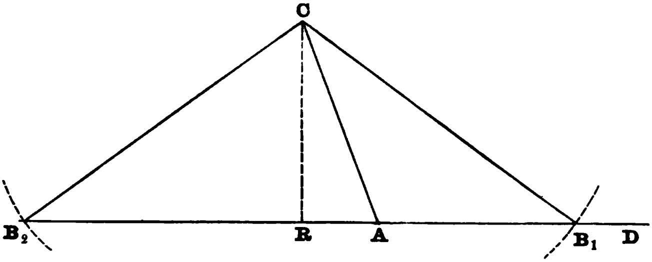

Fig. 12.

$$\begin{aligned}
\text{CR} &= b \sin (180^\circ - A) \\
\text{RA} &= b \cos (180^\circ - A) \\
\text{RB} &= \sqrt{a^2 - \text{CR}^2} = \sqrt{(a + \text{CR})\ (a - \text{CR})} \\
c &= \pm\ \text{RB} - \text{RA} \\
\text{Angle RCA} &= A - 90^\circ
\end{aligned}$$

$$\begin{aligned}
\sin \mathrm{RCB} &= \frac{\mathrm{RB}}{a}, \text{ or } \cos \mathrm{RCB} = \frac{\mathrm{CR}}{a} \\
C &= \pm \mathrm{RCB} - \mathrm{RCA} \\
B &= 180^\circ - (A + C).
\end{aligned}$$

74. EXAMPLE. Given $a = 65$, $b = 50$, and $A = 137^\circ\ 24'$.

$$\begin{aligned}
180^\circ - A &= 42^\circ\ 36' \\
\mathrm{CR} &= 50 \times 0.677 = 33.85 \\
\mathrm{RA} &= 50 \times 0.736 = 36.80 \\
\mathrm{RB} &= \sqrt{(65)^2 - (33.85)^2} = \pm\ 55.49 \\
c &= \pm\ 55.49 - 36.80 \\
c_1 &= 18.69 \\
[c_2 &= -92.29] \\
\mathrm{ACR} &= 137^\circ\ 24' - 90^\circ = 47^\circ\ 24' \\
\sin \mathrm{RCB} &= \frac{55.49}{65} = 0.854 \\
\mathrm{RCB} &= 58^\circ\ 40' \\
C_1 &= 58^\circ\ 40' - 47^\circ\ 24' = 11^\circ\ 16' \\
[C_2 &= -58^\circ\ 40' - 47^\circ\ 24 = -106^\circ\ 4'] \\
B_1 &= 180^\circ - (137^\circ\ 24' + 11^\circ\ 16') = 31^\circ\ 20' \\
[B_2 &= 180^\circ - (137^\circ\ 24' - 106^\circ\ 4') = 148^\circ\ 40']
\end{aligned}$$

75. SPECIAL CASES. (6) The arc drawn from **C**, Fig. 12, as a centre, and with a as a radius can cut the base of the triangle in only one point $\mathbf{B}_1$. The other point $\mathbf{B}_2$ is in the prolongation of the base. The triangle $\mathbf{AB}_2\mathbf{C}$ is never admissible as a solution because it never contains the given angle A as one of its parts. The triangle $\mathbf{AB}_1\mathbf{C}$ is admissible when, as in the example above given, the value of a is greater than that of b. The conditions of this special case are

$$A > 90^\circ \text{ and } a > b.$$

(7) If the given value of a were equal to that of b, the triangle $\mathbf{AB}_1\ \mathbf{C}$ would be merely a straight line, the

point $\mathbf{B}_1$ falling at **A**, and the triangle $\mathbf{AB}_2\mathbf{C}$ would be inadmissible as before. The conditions of this special case are

$$A > 90^\circ \text{ and } a = b.$$

(8) If the given value of a were less than that of b, both points, $\mathbf{B}_1$ and $\mathbf{B}_2$, would fall in the prolongation of the base, and both triangles would be *inadmissible;* or, a being shorter than the perpendicular, neither of the points $\mathbf{B}_1$ and $\mathbf{B}_2$ could be found, and both triangles would be *impossible.*

The conditions of this special case are

$$A > 90^\circ \text{ and } a < b.$$

76. Summary of Special Cases.

$A < 90^\circ$	$a < b \sin A$	(1). Both solutions impossible.
	$a = b \sin A$	(2). The two solutions identical.
	$b > a > b \sin A$	(3). Both solutions possible.
	$a = b$	(4). One solution an isosceles triangle, the other a straight line.
	$a > b$	(5). Both solutions possible, one admissible the other inadmissible.
$A > 90^\circ$	$a > b$	(6). Both solutions possible, one admissible the other inadmissible.
	$a = b$	(7). One solution a straight line, the other an inadmissible isosceles triangle.
	$a < b$	(8). Both solutions either inadmissible or impossible.

77. Exercises.

⁂ Before solving a triangle, examine the data to determine what solutions are possible and admissible. Article 76.

1. Given one side of a triangle 18 feet long, another 21 feet, and the angle opposite the former 49° 17′, to find the other parts.

2. Given one side of a triangle 47 feet long, another 37 feet, and the angle opposite the latter 50° 10′, to find the other parts.

3. Given one side of a triangle 83½ rods long, another 65¼ rods, and the angle opposite the latter 39° 52′, to find the other parts.

4. One angle of a triangle measures 56° 49′, the opposite side 25 inches, and one of the adjacent sides 28 inches. Find the other parts.

5. One angle of a triangle measures 75° 32′, the opposite side 82.5 feet, and one of the adjacent sides 84 feet. Find the other parts.

6. Given one side of a triangle 55 feet long, another 25 feet, and the angle opposite the latter 29° 15′, to find the other parts.

7. One angle of a triangle measures 72° 18′, the opposite side 57 feet, and one of the adjacent sides 49 feet. Find the other parts.

8. One angle of a triangle measures 32° 19′, the opposite side 12 feet, and another side 49 feet. Find the other parts.

9. Given two sides of a triangle, 92 and 103 feet long, and the angle opposite the greater side, 68° 20′, to find the other parts.

10. Given two sides of a triangle each 17 feet long and the angle opposite one of them 78°, to find the other parts.

11. Given one side of a triangle 41 feet long, another

24.969 feet, and the angle opposite the latter 37° 31′, to find the other parts.

12. One angle of a triangle measures 124° 37′, the opposite side 58, and another side 52 feet. Find the other parts.

13. Given two sides of a triangle, 102 and 121 feet long, and the angle opposite the latter, 142° 36′, to find the other parts.

14. One side of a triangle measures 57, another 100 feet, and the angle opposite the former 150°. Find the other parts.

15. One angle of a triangle measures 120°, an adjacent side 200 feet, and the opposite side 150 feet. Find the other parts.

16. Given two sides of a triangle, 5.32 and 6.54 metres, and the angle opposite the former, 32° 18′, to find the other parts.

17. One angle of a triangle measures 172° 15′, the opposite side 200 yards, and another side 100 yards. Find the third side.

18. Two sides of a triangle measure 15 yards and 100 yards, and the angle opposite the former 7° 45′. Find the other parts.

19. One angle of a triangle measures 100°, and the opposite side 100 feet; if one of the adjacent sides measures 50 feet, what must the other side measure?

20. From a given point outside of a circle two lines forming an angle of 25° are drawn, one through the centre of the circle, the other cutting the circumference in two points. If the distance from the given point to the centre of the circle is 15 inches, and the radius of the circle is 10 inches, what are the distances from the given point to the points where the circumference is cut by the second given line?

IV.

78. *Given the three sides of a triangle to find the angles.*

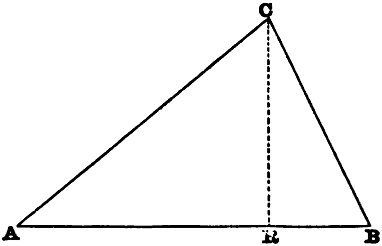

Fig. 13.

Let a, b, and c be the given parts. A perpendicular drawn to the longest side from the opposite vertex falls within the triangle. For, if c is the longest side, C is the largest angle, and the angles A and B must both be acute, so that the perpendicular from C falls between the points A and B, making

$$c = \mathrm{AR} + \mathrm{RB}.$$

From the right triangles ARC and BRC come the equations

$$b^2 = \overline{\mathrm{AR}}^2 + \overline{\mathrm{RC}}^2$$
$$a^2 = \overline{\mathrm{RB}}^2 + \overline{\mathrm{RC}}^2$$

whence, by subtraction,

$$b^2 - a^2 = \overline{\mathrm{AR}}^2 - \overline{\mathrm{RB}}^2$$

and, by factoring,

$$(b+a)\ (b-a) = (\mathrm{AR}+\mathrm{RB})\ (\mathrm{AR}-\mathrm{RB})$$

or

$$(b+a)\ (b-a) = c\ (\mathrm{AR}-\mathrm{RB})$$

whence

$$\mathrm{AR}-\mathrm{RB} = \frac{(b+a)\ (b-a)}{c}$$

by which AR − RB may be computed.

By taking half the sum of AR + RB and AR − RB, we get AR; and by taking half the difference, we get RB.

Then, to find the angles we have

$$\cos A=\frac{\mathrm{AR}}{b} \qquad \cos B=\frac{\mathrm{RB}}{a}$$

$$C=180^\circ-(A+B)$$

79. Example. Given $a=14$, $b=16$, and $c=20$.

$$\mathrm{AR}+\mathrm{RB}=20 \qquad \mathrm{AR}-\mathrm{RB}=\frac{30\times 2}{20}=3$$

$$\mathrm{AR}=\tfrac{1}{2}(20+3)=11.5 \qquad \mathrm{RB}=\tfrac{1}{2}(20-3)=8.5$$

$$\cos A=\frac{11.5}{16}=0.719 \qquad \cos B=\frac{8.5}{14}=0.607$$

$$A=44^\circ \qquad B=52^\circ\ 39'$$

$$C=83^\circ\ 21'.$$

80. Exercises.

1. What are the angles of a triangle of which the sides measure 15.3, 16.5, and 14.6 rods?

2. If the three sides of a triangle measure 100, 120, and 140 yards, what are the angles?

3. Find the angles of a triangle of which the sides measure 22, 28, and 48 feet.

4. Find the angles of a triangle of which the sides measure 9, 11, and 13 feet.

5. Find the angles of a triangle of which the sides measure 18, 22, and 26 feet.

6. Find the angles of a triangle of which the sides measure 45, 55, and 65 feet.

7. I have drawn on paper a triangle with sides 4.1 inches, 2.8 inches, and 3.3 inches long. What are the angles?

8. To find the angle formed by two sides of a field, I measure from the corner 10 rods along one side to a stake, and 12 rods along the other side to a stake; after which I measure the distance, 17.32 rods, between the two stakes. What is the size of the angle at the corner of the field?

LOGARITHMIC SOLUTION OF OBLIQUE TRIANGLES.

81. The most convenient logarithmic solutions will be given in a future chapter. (See Chapter VII.) But logarithms may be applied to the formulas of this chapter when closer approximations are desired than are attainable by three-place tables. For the purpose of comparison, some examples already solved with three-place natural functions are here solved with five-place logarithmic functions.

I.

82. Given $A=40° \ 20'$, $B=57° \ 30'$, and $b=76$ to find the other parts.

For the formulas, turn to page 36.

$$C=180°-(40° \ 20'+57° \ 30')=82° \ 10'.$$

log b =	1.88081	log b =	1.88081
log cos A =	9.88212	log sin A =	9.81106
log AR =	1.76293	log CR =	1.69187
AR =	57.934		
log cot B =	9.80419	log csc B =	0.07397
log CR =	1.69187	log CR =	1.69187
log RB =	1.49606	log a =	1.76584
RB =	31.337	a =	58.323
c =	89.271		

Compare these results with those of the same example on page 36.

II.

83. Given $A=31° \ 20'$, $b=34$, and $c=42$ to find the other parts.

For formulas turn to page 39.

$\log b$ =	1.53148	$\log b$ =	1.53148
$\log \cos A$ =	9.93154	$\log \sin A$ =	9.71602
log AR =	1.46302	log CR =	1.24750
AR =	29.041	colog RB =	8.88742
c =	42	$\log \tan B$ =	10.13492
RB =	12.959		
$\log \csc B$ =	0.09337	B =	53° 45′ 37″
log CR =	1.24750	C =	94° 54′ 23″
$\log a$ =	1.34087		
a =	21.922		

Compare these results with those of the same example on page 39.

III.

84. Given $a=37$, $b=47$, and $A=49° \; 45'$ to find the other parts.

For formulas turn to page 43.

$\log b$ =	1.67210	$\log b$ =	1.67210
$\log \cos A$ =	9.81032	$\log \sin A$ =	9.88266
log AR =	1.48242	log CR =	1.55476
AR =	30.3686	CR =	35.8725
		a =	37
$\log (a+\text{CR})$ =	1.86257	$a+$CR =	72.8725
$\log (a-\text{CR})$ =	0.05211	$a-$CR =	1.1275
	2)1.91468		
log RB =	0.95734	log RB =	0.95734
RB =	9.0644	colog a =	8.43180
c_1 =	39.4330	log sin RCB =	9.38914
c_2 =	21.3042	RCB =	14° 10′ 52″
		ACR = 90°$-A$ =	40° 15′
B_1 =	75° 49′ 8″	C_1 =	54° 25′ 52″
B_2 =	104° 10′ 52″	C_2 =	26° 4′ 8″

Compare these results with those of the same example on page 44.

IV.

85. Given $a=14$, $b=16$, and $c=20$ to find the angles. For formulas turn to page 54.

$$
\begin{array}{lllllll}
b+a & = & 30 & \log(b+a) & = & 1.47712\\
b-a & = & 2 & \log(b-a) & = & 0.30103\\
c & = & 20 & \operatorname{colog} c & = & \underline{8.69897}\\
\mathrm{AR}-\mathrm{RB} & = & 3 & \log(\mathrm{AR}-\mathrm{RB}) & = & 0.47712\\
\mathrm{AR}+\mathrm{RB} & = & 20 & & & \\
\mathrm{AR} & = & 11.5 & \mathrm{RB} & = & 8.5\\
\log \mathrm{AR} & = & 1.06070 & \log \mathrm{RB} & = & 0.92942\\
\operatorname{colog} b & = & \underline{8.79588} & \operatorname{colog} a & = & \underline{8.85387}\\
\log\cos A & = & 9.85658 & \log\cos B & = & 9.78329\\
A & = & 44^\circ\, 2'\, 55'' & B & = & 52^\circ\, 37'\, 0''
\end{array}
$$

$$C = 83^\circ\, 20'\, 5''$$

Compare these results with those of the same example on page 55.

86. Exercises.

1. Find the other parts of a triangle of which one side measures 4283.6 feet, one adjacent angle 67° 16′ 47″, and the other 58° 48′ 17″.

2. Two angles of a triangle measure 38° 18′ 45″ and 109° 42′ 35″, and the shortest side 592.3 metres. Find the other two sides.

3. Two sides of a triangle measure 49.85 feet and 51.92 feet, and the angle opposite the former 53° 47′ 20″. Find the other parts.

4. Find the other parts of a triangle of which two sides measure 48.26 and 52.88 feet, and the included angle 62° 10′ 25″.

5. Two angles of a triangle measure 87° 25′ 15″ and 50° 32′ 19″; and the longest side 489.38 feet. Find the other two sides.

6. Two sides of a triangle measure 327.9 feet and 296.8 feet, and the angle opposite the former 88° 19′ 35″. Find the other parts.

7. Find the angles of a triangle of which the sides measure 263.47, 289.41, and 324.19 rods.

8. Find the other parts of a triangle of which two sides measure 328.68 and 329.48 metres, and the included angle 89° 14′ 35″.

9. Two sides of a triangle measure 437.68 yards and 897.54 yards, and the angle opposite the latter 175° 28′ 40″. Find the other parts.

10. Find the angles of a triangle of which the sides measure 583.9, 492.3, and 621.4 feet.

11. Two sides of a triangle measure 597.3 and 1269.4 feet, and the included angle 171° 24′ 16″. Find the other parts.

12. Find the angles of a triangle of which the sides measure 1231, 697, and 705 feet.

13. Two sides of a triangle measure 496 feet and 497 feet and the angle opposite the former 2° 23′ 15″. Find the other parts.

14. Find the angles of a triangle of which the sides measure 1001, 1002, and 1003 feet.

15. Two sides of a triangle measure 537.23 and 537.67 rods, and the included angle 96° 18′ 25″. Find the other parts.

16. Two sides of a triangle are in the ratio of 1 to 0.993465, and the included angle is 115°. Find the other angles, and the ratio of the third side to each of the others.

17. The three sides of a triangle measure 3484, 3277, and 3158 feet. Find the angles, and find the lengths of the perpendiculars drawn from each corner to the opposite side.

18. To find the angle made by two roads, I pace along one road 237 paces, then across the fields to the other road 209 paces, and then by the other road 253 paces back to the starting-point. What angle do the two roads make?

87. Miscellaneous Exercises.

1. From a given point to a given straight line are drawn a perpendicular and two oblique lines. The latter are 10 and 15 inches long, and are both on the same side of the perpendicular. The distance between the points where they meet the straight line is 8 inches. Find the length of the perpendicular.

2. The data remaining the same as in the preceding exercise, suppose the oblique lines had been drawn on different sides of the perpendicular, what would then have been the distance between the points where they met the straight line?

3. From a belfry 148 feet above the ground the angles of depression of the top and bottom of a flagstaff are found to be 38° 18′ and 46° 38′; and the flagstaff stands on a mound 20 feet higher than the ground under the belfry. How high is the flagstaff?

4. The horizontal distance between two stations, A and B, is 700 feet; but B is 50 feet below the level of A. In a line with A and B, and beyond B looking from A, stands a tower whose height is to be found. The angle of elevation of its top at A is 20° 51′; and at B, 31° 26′. How high is the tower above the level of B, and what is its horizontal distance from B?

5. The diagonals of a parallelogram form an angle of 50° with each other, and their lengths are 22 and 42 inches. Find the sides and angles of the parallelogram.

6. The sides of a triangle are $AB = 26$ feet, $BC = 28$ feet, and $CA = 32$ feet. Within the triangle is a point 15 feet from A and 16 feet from B. How far is this point from C?

7. The triangle described in the preceding exercise being laid out on level ground, an upright pole 40 feet high is planted at a point 15 feet from A and 16 feet from B. Cords are stretched from the top of the pole to the three corners of the triangle and fastened there. What are the lengths of these cords, and what angles do they make with one another?

8. A and B are two stations whose distance apart is known; C and D are two other stations whose distance apart is to be found by means of the angles measured at A and at B.

Given: Distance A B = 3.42 miles.
Angles, B A D = 32° 17′.
D A C = 44° 28′.
D B C = 63° 12′.
C B A = 46° 48′.

To find C D.

Draw the figure to a scale of an inch to a mile.

9. Having computed the distance C D, as in the preceding exercise, we can now find the distance between two more stations, E and F, by means of angles measured at C and at D.

Given: Angles, D C F = 35° 44′.
F C E = 61° 18′.
F D E = 59° 37′.
E D C = 40° 2′.

To find E F.

Draw the figure to a scale of an inch to a mile.

⁂ The last two exercises exemplify the method of surveying known as triangulation. A base line in some suitable place is very accurately measured, and stations for measuring angles are selected at convenient points (usually hill-tops) all over the territory to be surveyed. Two of these stations, of course, are the two ends of the base line. The lines drawn from station to station form a net-work of triangles, whose angles are measured at the different stations, but whose sides are to be computed from these measured angles and the known length of the base line. The triangles given in Art. 88 were taken from a survey of this kind.

10. At just eight o'clock two light-houses were observed from the deck of a ship, one bearing N. 46° W., and the other N. 7° W. At nine o'clock the light-houses were again observed, the first bearing S. 56° W., and the second N. 77° W. The chart shows that the distance between the light-houses is 12.3 miles, and the bearing of the second from the first is N. 32° E. How far did the ship sail during the hour, supposing she held a straight course?

11. A and B are two small islands in the harbor. From the top of a monument 210 feet above the water level, the angle of depression of A is found to be 10° 17′; and that of B, 12° 42′, and the horizontal angle* between them 57° 39′. How far apart are the two islands, and what is the horizontal distance of each from the monument?

12. From the top of a tower 180 feet high standing within a level triangular field, the angles of depression and the bearings of the three corners, A, B, and C, are taken as follows:

A	bears	N. 52° W.,	depressed	12° 45′.
B	"	S. 81° E.,	"	9° 15′.
C	"	S. 43° W.,	"	10° 30′.

Find the sides and angles of the field, and the position of the tower within it.

13. A surveyor measuring a line in a northerly direction comes to an impassable swamp, and, having no instrument with which to measure angles, proceeds to measure with his chain (four rods long), as follows: From his station A south of the swamp, a line A B in a northeasterly direction 12 chains long; from station A, a line A C in a northwesterly direction 15 chains long; from B to C, 15.71 chains. The line B C lies beyond the swamp, and is crossed at a point D by the line running northerly from A. By measuring B D = 6.08 chains or D C = 9.63 chains, the surveyor has data sufficient to compute the length of the line A D which crosses the swamp. Find A D.

---

* The horizontal angle is the angle formed by two lines proceeding from the point of observation horizontally to points directly over (or under) the objects observed.

14. To determine the position of a point by means of the bearings taken from it of three other points whose positions are known.

To give the problem a practical form, let the point to be determined be the position of a boat in a harbor, and the three known points three observable objects, A, B, and C, on the shore, whose distances apart are known; namely, A B = 520, B C = 602, and C A = 410 yards. The observer in the boat sees A directly before him, B to the left, and C to the right; but A is farther off from him than either B or C. Calling his place of observation O, he measures the two angles A O B = 21° 40′, and A O C = 16° 20′. The points A, B, and C having been laid down (on a chart), the point O can be laid down by the following construction: On B C as a base, construct a triangle B D C making the angle D C B equal to 21° 40′ (A O B), and the angle D B C equal to 16° 20′ (A O C). Observe that the triangles B A C and B D C have the same base, and that the vertices, A and D, are on the same side of it. Draw a circle through the three points B, D, and C. Through A and D draw a straight line. This line cuts the circumference of the circle at D, and, if prolonged, at another point. This other point is O, whose position was to be found. The reason for this construction lies in the two facts, geometrically evident, that the angles D C B and A O B are equal, being inscribed in the same segment of the circle, and that the angles D B C and A O C are equal for a similar reason. The distances

A O, B O, and C O can be computed trigonometrically by solving first the triangle A B C, then B D C, then A B D or A C D, and finally A O B and A O C. With the data as above given, compute these three distances.

88. Examples of Triangles.

	SIDES.	ANGLES.		SIDES.	ANGLES.
1.	33296	36° 51′ 22″	*10.*	46963	42° 48′ 12″
	44477	53° 14′ 51″		68622	83° 8′ 54″
	55511	89° 53′ 47″		55950	54° 2′ 54″
2.	51589	99° 31′ 36″	*11.*	68622	84° 13′ 20″
	44477	58° 14′ 20″		36686	32° 8′ 1″
	19794	22° 14′ 4″		61803	63° 38′ 39″
3.	49642	62° 40′ 11″	*12.*	36686	47° 58′ 30″
	55511	96° 35′ 3″		42091	58° 27′ 44″
	19794	20° 44′ 46″		47367	73° 33′ 46″
4.	44477	93° 33′ 30″	*13.*	36686	44° 9′ 21″
	38233	59° 5′ 16″		24048	27° 10′ 13″
	20476	27° 21′ 14″		49891	108° 40′ 26″
5.	44477	67° 45′ 33″	*14.*	42091	113° 13′ 1″
	18138	22° 10′ 36″		24048	31° 40′ 22″
	48052	90° 3′ 51″		26342	35° 6′ 37″
6.	38233	88° 58′ 27″	*15.*	47367	69° 3′ 36″
	18138	28° 18′ 57″		49891	79° 38′ 53″
	33983	62° 42′ 36″		26342	31° 17′ 31″
7.	20476	21° 12′ 53″	*16.*	26342	60° 40′ 6″
	48052	121° 52′ 28″		17590	35° 36′ 6″
	33983	36° 54′ 39″		30035	83° 43′ 48″
8.	33983	63° 46′ 38″	*17.*	30035	45° 45′ 41″
	37465	98° 30′ 21″		21991	31° 38′ 19″
	11528	17° 43′ 1″		40913	102° 36′ 0″
9.	37465	49° 54′ 7″	*18.*	21991	41° 12′ 38″
	46963	73° 30′ 37″		29784	63° 9′ 49″
	40883	56° 35′ 16″		32334	75° 37′ 33″

19.	40913	105° 37′ 23″
	29784	44° 30′ 52″
	21153	29° 51′ 45″
20.	30035	64° 24′ 38″
	32334	76° 9′ 18″
	21153	39° 26′ 4″
21.	29784	67° 7′ 37″
	31209	74° 53′ 42″
	19892	37° 58′ 41″
22.	31209	51° 44′ 17″
	28639	46° 5′ 54″
	39376	82° 9′ 49″
23.	28639	33° 40′ 17″
	38898	48° 51′ 15″
	51216	97° 28′ 28″
24.	38898	85° 28′ 45″
	28730	47° 25′ 2″
	28585	47° 6′ 13″
25.	28730	35° 15′ 35″
	20436	24° 14′ 39″
	42883	120° 29′ 46″
26.	28585	100° 16′ 33″
	20436	44° 42′ 17″
	16671	35° 1′ 10″
27.	38898	65° 1′ 10″
	42883	92° 7′ 12″
	16671	22° 51′ 38″
28.	20436	30° 4′ 36″
	22104	32° 49′ 27″
	36301	117° 5′ 57″
29.	22104	37° 23′ 54″
	34827	73° 7′ 31″
	34084	69° 28′ 35″
30.	34827	86° 45′ 41″
	30168	59° 51′ 53″
	19189	33° 22′ 26″
31.	19189	41° 47′ 41″
	24143	56° 59′ 4″
	28455	81° 13′ 15″
32.	24143	65° 21′ 13″
	25946	77° 37′ 39″
	15993	37° 1′ 8″
33.	15993	35° 10′ 10″
	23448	57° 37′ 3″
	27733	87° 12′ 47″
34.	28476	81° 27′ 12″
	20014	44° 1′ 47″
	23448	54° 31′ 1″
35.	20014	48° 3′ 30″
	26268	77° 29′ 25″
	21892	54° 27′ 5″
36.	8688	38° 37′ 6″
	13916	88° 35′ 16″
	11087	52° 47′ 38″
37.	8688	18° 13′ 39″
	21303	50° 4′ 57″
	25809	111° 41′ 24″
38.	13916	40° 10′ 18″
	21303	80° 55′ 58″
	18471	58° 53′ 44″
39.	11087	21° 56′ 38″
	25809	119° 33′ 5″
	18471	38° 30′ 17″
40.	25809	68° 43′ 41″
	26179	70° 56′ 56″
	17922	40° 19′ 23″

CHAPTER IV.

A GENERAL VIEW OF TRIGONOMETRIC FUNCTIONS.

89. Hitherto, our treatment of trigonometric functions has been restricted to those of acute angles. Even obtuse angled triangles were solved by using, not the obtuse angle, but its supplement, which was an acute angle. But a general treatment of our subject requires that all restrictions on the magnitude of angles be removed; and that we consider the functions, not only of acute angles, but of obtuse angles, and also of angles exceeding two, or three, or four, or any number of right angles.

ANGLES.

90. To form a conception of an angle as defined in trigonometry, we may regard one side as holding a fixed position, and the other a movable one. The movable side is supposed to have generated the angle by starting from the fixed position of the first side and turning upon the vertex as upon a pivot to the position of the second side. The angle thus generated is greater or less according to the amount of turning. If the amount of turning is less than a quarter of a revolution, the angle generated is an acute angle, Fig 14; if the turning amounts to more

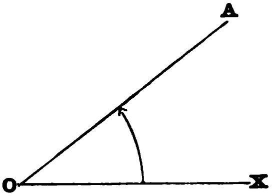

Fig. 14.

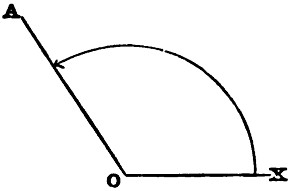

Fig. 15.

than a quarter and less than half of a revolution, the angle generated is an obtuse angle, Fig. 15; and if the turning amounts to more than half a revolution, the angle generated exceeds two right angles, as in Figs. 16 and 17.

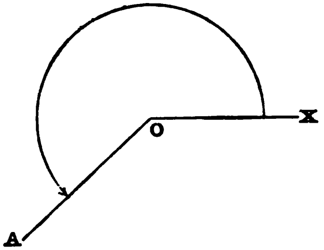

Fig. 16.

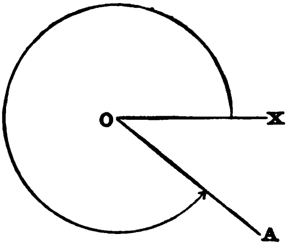

Fig. 17.

Such angles as the last two are called convex angles. They are excluded from consideration in elementary geometry, but are taken into account in trigonometry. When the revolving line **O A** has made one entire revolution, returning to its initial position **O X**, the angle generated equals four right angles, or 360 degrees. If the turning continues still further, the angle generated exceeds four right angles, or 360°; and, as the turning may be endless, the number of right angles, or the number of degrees, in an angle is unlimited. This leads to the enlarged conception and definition of an angle as used in trigonometry. *An angle is any amount of turning in a fixed plane about a point that will carry a straight line passing through that point from any direction to any other direction.*

91. Since we may regard **O A**, Figs. 14, 15, 16, 17, as having made any number of entire revolutions before making the partial revolution indicated by the curved arrow, the trigonometric angle **X O A** has an unlimited number of particular values. Thus it differs from the geometric angle **XOA**, which has but one value. For example, in Fig. 14, if the value of the geometric angle **XOA** is 30°, the value of the trigonometric angle is 30°, or 390°, or 750°, or any one of an unlimited series ot values each differing from the one next to it by 360°. And in general, if A represents any one value of an angle, all its values are represented by the expression

$$A \pm k\,360°$$

where k is any integral number (including 0).

The *inital side* of an angle is the side which is regarded as fixed in position; and the *terminal side* is the revolving or generating line in its final position.

92. If two lines **XX′** and **YY′**, Fig. 18, are drawn perpendicular to each other and **OX** is taken as the initial side of all angles generated by the line **OA** revolving in the same plane with the two lines about the point **O**, the angles are classified by noting in which of the four right angles **XOY**, **YOX′**, **X′OY′**, or **Y′OX** their terminal sides lie. The lines **XX′** and **YY′** are called axes; and they divide the plane into four quadrants, the first quadrant **XOY**, the second **YOX′**, the third **X′OY′**, and the fourth **Y′OX**. Angles are said

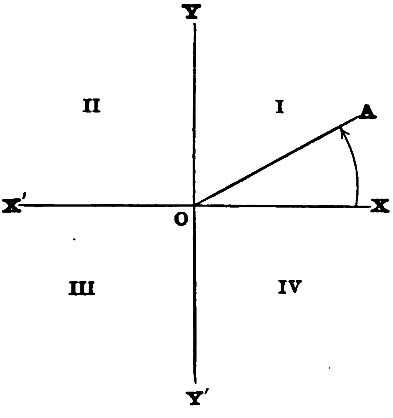

Fig. 18.

to be *of* or *in* the first, second, third or fourth quadrant when their terminal sides lie in those quadrants respectively. Thus, all acute angles are angles of the first quadrant; as are also all angles formed by adding to any acute angle an integral number of times 360°. All obtuse angles are angles of the second quadrant; as are also all angles formed by adding to any obtuse angle an integral number of times 360°. Convex angles, and those formed from them by adding an integral number of times 360° are angles of the third or of the fourth quadrant.

93. Angles are regarded as positive or negative according to the way the generating line is supposed to have turned about the vertex. Positive turning is "against the sun," or contrary to the usual motion of clock-hands. It would carry the generating line **OA** from **OX** to **OY**, thence to **OX′**, thence to **OY′**, and thence to **OX** again. Negative turning would carry the line around the other way. A positive angle is generated by a line turning positively, and a negative angle by one turning negatively.

94. The way the generating line is supposed to have turned is indicated by the order in which the letters denoting the angle are written. Thus **XOA** indicates that the generating line turned from **OX** to **OA**, while **AOX** would indicate that it turned the contrary way, making **AOX** a negative angle, if **XOA** is a positive one.

Hence the same angle is represented as positive or negative according to the way it is written, **AOX** being the negative of **XOA**, so that

$$\mathrm{XOA}+\mathrm{AOX}=0 \quad . \; . \; . \; . \; . \quad [39]$$

NOTE. Formula [39] is only a particular case of a more general formula. For if both angles XOA and AOX be read positively, or both negatively, or one positively and the other negatively, but with their general values (Art. 91), the result will be

$$\mathrm{XOA}+\mathrm{AOX}=\pm k\cdot 360^\circ.$$

94. By attending to the distinction between positive and negative angles, we can represent geometrically the negative complement of an obtuse angle, and the negative supplement of a convex angle. Thus in Fig. 19 the complement of the obtuse angle **XOA** is the negative angle **AOY**. For the positive turning from **OX** to **OA** followed by the negative turning from **OA** to **OY** is equivalent to the one positive turning from **OX** to **OY**. Hence in whatever quadrant **OA** may lie it is always true that

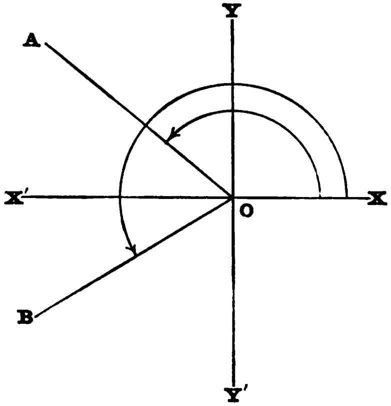

Fig. 19.

$$XOA+AOY=XOY=90° \quad . \quad . \quad . \quad . \quad . \quad [40]$$

Also the supplement of the positive angle **XOB**, Fig. 19, is the negative angle **BOX′**; so that in whatever quadrant **OB** may lie it is always true that

$$XOB+BOX'=XOX'=180° \quad . \quad . \quad . \quad . \quad [41]$$

POSITIVE AND NEGATIVE LINES.

95. Straight lines are regarded as positive or negative according to the way a point is supposed to have moved in generating them. Every line is regarded as having a beginning, called its *origin*, and an end, called its *termination;* and these are indicated by letters, the letter denoting the origin always being written first. Thus **AB** denotes a line generated by a point moving from **A** to **B**, while **BA** denotes a line generated by a point moving the opposite way. If the line **AB** is regarded as positive the line **BA** must be regarded as negative; so that it is always true that

$$AB+BA=0 \quad \ldots\ldots \quad [42]$$

96. If there are three points **A, B,** and **C** arranged in any order in a straight line, Fig. 20, it is always true that the movement of a point from **A** to **B** followed by its movement from **B** to **C** leaves the point just as far from **A** as does one movement from **A** to **C.** The general statement of this fact is this, that the algebraic sum of the distances **AB** and **BC** is always **AC,** whatever be the order of the points **A, B,** and **C** in the line. This is expressed by the equation

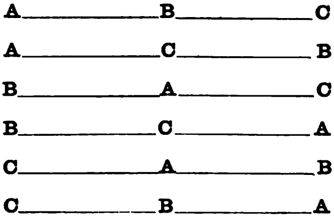

Fig. 20.

$$AB+BC=AC \quad \ldots\ldots \quad [43]$$

which by adding **CA** to each member becomes

$$AB+BC+CA=0 \quad \ldots\ldots \quad [44]$$

COÖRDINATES.

97. The position of any point in the plane of the axes **XX′** and **YY′**, Fig. 21, is fully known if its distance from each axis is given together with an indication as to which way from the axis the distance is to be measured. The distance from each axis is measured on, or in a direction parallel to the other. It is adopted as a rule that all distances measured from left to right, or in the direction **X′X**, shall be expressed in positive numbers. Then distances measured from right to left, or in the direction **XX′**, will be expressed in negative numbers. The same distinction is made between distances measured upward, or in the direction **Y′Y**, and those measured downward, or in the direction **YY′**, the former being expressed in positive, the latter in negative, numbers. The distance of a point from the axis **YY′** to the right or left is called its *abscissa*, and its distance above or below the axis **XX′** is called its *ordinate*, and both distances, when spoken of together, are called its *coördinates*. The position of a point, then, is fully known if its coördinates are given. Thus, Fig. 21, if the abscissa of a point is +4 and the ordinate +3, the point is found by laying off **OM** = +4 and **ON** = +3, and drawing through **M** and **N** lines parallel to the axes. These lines intersect in **A**, the point to be found. In the same way, the point **B** is found from its coördinates, −3 and +5; the point **C** from its coördinates, −4 and −4; and the point **D** from its coördinates, +5 and −2. The general symbols for the coördinates of a point are x and y, the former for the abscissa, and the latter for the ordinate; so that the position of points may be briefly expressed by writing pairs of equations, thus:

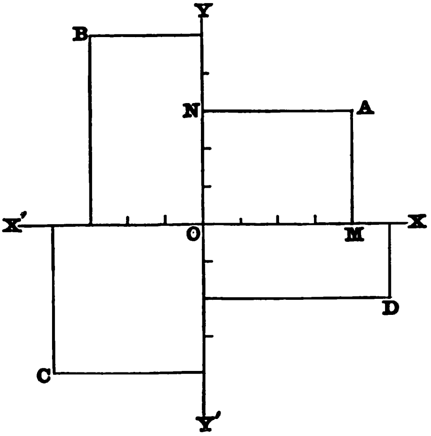

Fig. 21.

$$\text{A, }\begin{cases} x=+4 \\ y=+3 \end{cases} \quad \text{B, }\begin{cases} x=-3 \\ y=+5 \end{cases} \quad \text{C, }\begin{cases} x=-4 \\ y=-4 \end{cases} \quad \text{D, }\begin{cases} x=+5 \\ y=-2 \end{cases}$$

or still more briefly thus:

A (4, 3), B (−3, 5), C (−4, −4), D (5, −2)

In such expressions the abscissa is always written first.

GENERAL DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS.

98. Let A denote an angle of any magnitude, positive or negative, of which the initial line is **O X** and the terminal line is **O A**. The terminal line may lie in any one of the four quadrants, as represented in Figs. 22, 23, 24, 25.

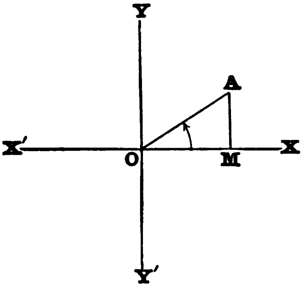

Fig. 22.

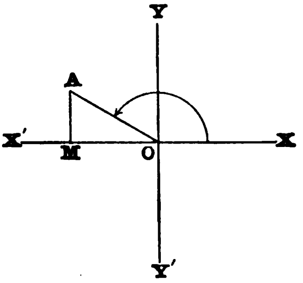

Fig. 23.

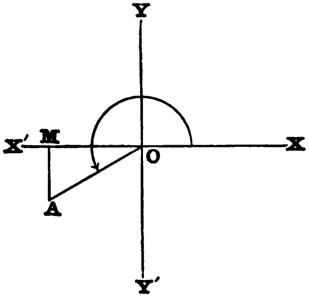

Fig. 24.

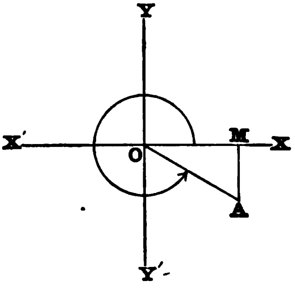

Fig. 25.

The point A is taken at any chosen distance, O A denoted by r, from the vertex of the angle; and the coördinates of A, are denoted by x and y.

Then the six functions of A are thus defined:

$$\left.\begin{aligned}
\sin A &= \frac{\text{ordinate}}{\text{distance}} = \frac{\text{MA}}{\text{OA}} = \frac{y}{r}\\
\tan A &= \frac{\text{ordinate}}{\text{abscissa}} = \frac{\text{MA}}{\text{OM}} = \frac{y}{x}\\
\sec A &= \frac{\text{distance}}{\text{abscissa}} = \frac{\text{OA}}{\text{OM}} = \frac{r}{x}\\
\cos A &= \frac{\text{abscissa}}{\text{distance}} = \frac{\text{OM}}{\text{OA}} = \frac{x}{r}\\
\cot A &= \frac{\text{abscissa}}{\text{ordinate}} = \frac{\text{OM}}{\text{MA}} = \frac{x}{y}\\
\csc A &= \frac{\text{distance}}{\text{ordinate}} = \frac{\text{OA}}{\text{MA}} = \frac{r}{y}
\end{aligned}\right\} \quad . \; . \; . \quad [45]$$

POSITIVE AND NEGATIVE VALUES OF THE FUNCTIONS.

99. The functions of angles in the different quadrants have positive or negative values dependent upon the values of the coördinates of the point A. The distance r is always positive, in whatever direction O A may point. The abscissa x is, Art. 97, positive when O A is in the first or fourth quadrant, and negative when it is in the second or third. The ordinate y is positive when O A is in the first or second quadrant, and negative when it is in the third or fourth.

Hence,

Of angles in the *first* quadrant, all the functions are positive.

Of angles in the *second* quadrant, the sine and cosecant are positive, the other functions are negative.

Of angles in the *third* quadrant, the tangent and cotangent are positive, the other functions are negative.

Of angles in the *fourth* quadrant, the secant and cosine are positive, the other functions are negative.

100. As an aid to the memory, Figs. 26, 27, and 28, show, by means of the signs + and —, in which quadrants the several functions are positive, and in which negative.

SINE AND COSECANT.

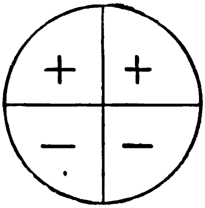

Fig. 26.

TANGENT AND COTANGENT.

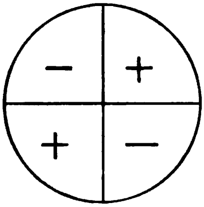

Fig. 27.

SECANT AND COSINE.

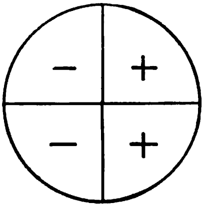

Fig. 28.

LIMITING VALUES OF THE FUNCTIONS.

101. If we suppose the line **OA**, Fig. 22, starting from the position **OX** to make one entire positive revolution, the angle **XOA**, denoted by A, will pass through all values from 0° to 360°, and each function will vary between certain limiting values, which may be deduced from [45].

102. At the start $A=0°$, and the terminal line **OA** coincides with **OX**; so that the coördinates of the point **A** are

$$x=r \quad \text{and} \quad y=0,$$

and [45] gives

$$\sin 0°=\frac{0}{r}=0 \qquad \cos 0°=\frac{r}{r}=1$$

$$\tan 0°=\frac{0}{r}=0 \qquad \cot 0°=\frac{r}{0}=\infty$$

$$\sec 0°=\frac{r}{r}=1 \qquad \csc 0°=\frac{r}{0}=\infty$$

103. When $A = 90°$, the terminal line **O A** coincides with **O Y**; so that the coördinates of the point **A** are

$$x = 0 \quad \text{and} \quad y = r,$$

and [45] gives

$$\sin 90° = \frac{r}{r} = 1 \qquad \cos 90° = \frac{0}{r} = 0$$

$$\tan 90° = \frac{r}{0} = \infty \qquad \cot 90° = \frac{0}{r} = 0$$

$$\sec 90° = \frac{r}{0} = \infty \qquad \csc 90° = \frac{r}{r} = 1$$

104. When $A = 180°$, the terminal line **O A** coincides with **O X′**; so that the coördinates of the point **A** are

$$x = -r \quad \text{and} \quad y = 0,$$

and [45] gives

$$\sin 180° = \frac{0}{r} = 0 \qquad \cos 180° = \frac{-r}{r} = -1$$

$$\tan 180° = \frac{0}{-r} = 0 \qquad \cot 180° = \frac{-r}{0} = \infty$$

$$\sec 180° = \frac{r}{-r} = -1 \qquad \csc 180° = \frac{r}{0} = \infty$$

105. When $A = 270°$, the terminal line **O A** coincides with **O Y′**; so that the coördinates of the point **A** are

$$x = 0 \quad \text{and} \quad y = -r,$$

and [45] gives

$$\sin 270° = \frac{-r}{r} = -1 \qquad \cos 270° = \frac{0}{r} = 0$$

$$\tan 270° = \frac{-r}{0} = \infty \qquad \cot 270° = \frac{0}{-r} = 0$$

$$\sec 270° = \frac{r}{0} = \infty \qquad \csc 270° = \frac{r}{-r} = -1$$

106. When $A = 360°$, the terminal line **O A** again coincides with **O X**, and the coördinates of the point **A** are again $x=r$ and $y=0$; so that the functions of 360° are identical in value with those of 0°.

107. If A should further increase from 360° to 720°, the terminal line would again pass through the same series of positions, and the functions would have the same limiting values in the same order of succession.

VARIATION OF THE FUNCTIONS BETWEEN THEIR LIMITING VALUES.

108. As the angle A increases from 0° to 360° the coördinates of the point A vary, but the distance r remains constantly the same. The effect of these changes in x and y upon the value of each function may be traced by reference to [45].

109. The SINE $\frac{y}{r}$, having a constant denominator, varies directly as the numerator. In the first quadrant, therefore, the sine increases from 0 to $+1$; in the second it decreases from $+1$ to 0; in the third it decreases from 0 to -1; and in the fourth it increases from -1 to 0.

At 180° the sine changes from having a positive to having a negative value; but at 360°, the change is the other way. These changes are indicated by writing

$$\sin 180° = \pm 0 \qquad \sin 360° = \mp 0,$$

the double sign showing which way the change takes place.

110. The TANGENT $\frac{y}{x}$ varies directly as the numerator and inversely as the denominator. As the angle passes from 0° to 90° y increases and x decreases, both of which changes increase the tangent, and the tangent passes through all positive values from 0 to $+\infty$.

As the angle passes from 90° to 180° the numerator decreases and the denominator arithmetically increases, both of which changes decrease the tangent arithmetically from an infinite value to 0. But in the second quadrant the tangent is negative, so that its arithmetical decrease is an algebraic increase. Algebraically speaking, then, the tangent *increases* in the second quadrant from $-\infty$ to 0. In the third quadrant, since x and y are both negative, the tangent is positive, and, as in the first quadrant, increases from 0 to $+\infty$. In the fourth quadrant, as in the second, the tangent decreases arithmetically from an infinite value to 0, but being negative its arithmetical decrease is an algebraic increase from $-\infty$ to 0.

111. It is to be noticed here that when a function changes from having a positive value in one quadrant to having a negative value in the next, or the reverse, the value of the function is either 0 or ∞. The tangent illustrates all these changes. Thus, at 90° it changes from positive to negative values, passing through the value ∞; at 180° from negative to positive, passing through the value 0; at 270° from positive to negative, passing through the value ∞; and at 360°, from negative to positive, passing through the value 0. These changes are indicated by means of the double sign, thus:

$$\tan 90° = \pm \infty \qquad \tan 270° = \pm \infty$$
$$\tan 180° = \mp 0 \qquad \tan 360° = \mp 0$$

112. The SECANT $\frac{r}{x}$, having a constant numerator, varies inversely as the denominator; but when the function is negative its arithmetical increase (or decrease) is an algebraic decrease (or increase). In the first quadrant it increases from $+1$ to $+\infty$; in the second it increases from $-\infty$ to -1; in the third it decreases from -1 to

$-\infty$; and in the fourth it decreases from $+\infty$ to $+1$. The changes at 90° and at 270° are thus indicated:

$$\sec 90^\circ = \pm\infty \qquad \sec 270^\circ = \mp\infty$$

113. The COSINE $\frac{x}{r}$, having a constant denominator, varies directly as the numerator. In the first quadrant the cosine decreases from $+1$ to 0; in the second it decreases from 0 to -1; in the third it increases from -1 to 0; and in the fourth it increases from 0 to $+1$. The changes at 90° and 270° are thus indicated:

$$\cos 90^\circ = \pm 0 \qquad \cos 270^\circ = \mp 0$$

114. The COTANGENT $\frac{x}{y}$ varies directly as the numerator and inversely as the denominator. In the first quadrant it decreases from $+\infty$ to 0; in the second it decreases from 0 to $-\infty$; in the third it decreases from $+\infty$ to 0; and in the fourth it decreases from 0 to $-\infty$. The changes at 90°, 180°, etc., are thus indicated:

$$\cot 90^\circ = \pm 0 \qquad \cot 270^\circ = \pm 0$$
$$\cot 180^\circ = \mp\infty \qquad \cot 360^\circ = \mp\infty$$

115. The COSECANT $\frac{r}{y}$, having a constant numerator, varies inversely as the denominator. In the first quadrant it decreases from $+\infty$ to $+1$; in the second it increases $+1$ to $+\infty$; in the third it increases from $-\infty$ to -1; and in the fourth it decreases from -1 to $-\infty$. The changes at 180° and 360° are thus indicated:

$$\csc 180^\circ = \pm\infty \qquad \csc 360^\circ = \mp\infty$$

116. The results of the foregoing discussion appear clearly in the table on the next page, which gives the value of each function for every tenth degree from 0° to 360°.

TABLE II.

FUNCTIONS OF ANGLES IN ALL QUADRANTS.

ANGLE.	SIN.	TAN.	SEC.	COS.	COT.	CSC.
0°	∓0	∓0	+1.000	+1.000	∓∞	∓∞
10	+0.174	+0.176	+1.015	+0.985	+5.671	+5.759
20	+0.342	+0.364	+1.064	+0.940	+2.747	+2.924
30	+0.500	+0.577	+1.155	+0.866	+1.732	+2.000
40	+0.643	+0.839	+1.305	+0.766	+1.192	+1.556
50	+0.766	+1.192	+1.556	+0.643	+0.839	+1.305
60	+0.866	+1.732	+2.000	+0.500	+0.577	+1.155
70	+0.940	+2.747	+2.924	+0.342	+0.364	+1.064
80	+0.985	+5.671	+5.759	+0.174	+0.176	+1.015
90°	+1.000	±∞	±∞	±0	±0	+1.000
100	+0.985	−5.671	−5.759	−0.174	−0.176	+1.015
110	+0.940	−2.747	−2.924	−0.342	−0.364	+1.064
120	+0.866	−1.732	−2.000	−0.500	−0.577	+1.155
130	+0.766	−1.192	−1.556	−0.643	−0.839	+1.305
140	+0.643	−0.839	−1.305	−0.766	−1.192	+1.556
150	+0.500	−0.577	−1.155	−0.866	−1.732	+2.000
160	+0.342	−0.364	−1.064	−0.940	−2.747	+2.924
170	+0.174	−0.176	−1.015	−0.985	−5.671	+5.759
180°	±0	∓0	−1.000	−1.000	∓∞	±∞
190	−0.174	+0.176	−1.015	−0.985	+5.671	−5.759
200	−0.342	+0.364	−1.064	−0.940	+2.747	−2.924
210	−0.500	+0.577	−1.155	−0.866	+1.732	−2.000
220	−0.643	+0.839	−1.305	−0.766	+1.192	−1.556
230	−0.766	+1.192	−1.556	−0.643	+0.839	−1.305
240	−0.866	+1.732	−2.000	−0.500	+0.577	−1.155
250	−0.940	+2.747	−2.924	−0.342	+0.364	−1.064
260	−0.985	+5.671	−5.759	−0.174	+0.176	−1.015
270°	−1.000	±∞	∓∞	∓0	±0	−1.000
280	−0.985	−5.671	+5.759	+0.174	−0.176	−1.015
290	−0.940	−2.747	+2.924	+0.342	−0.364	−1.064
300	−0.866	−1.732	+2.000	+0.500	−0.577	−1.155
310	−0.766	−1.192	+1.556	+0.643	−0.839	−1.305
320	−0.643	−0.839	+1.305	+0.766	−1.192	−1.556
330	−0.500	−0.577	+1.155	+0.866	−1.732	−2.000
340	−0.342	−0.364	+1.064	+0.940	−2.747	−2.924
350	−0.174	−0.176	+1.015	+0.985	−5.671	−5.759
360°	∓0	∓0	+1.000	+1.000	∓∞	∓∞

FORMULAS GENERALIZED.

117. Formulas [3] to [10], expressing the relations between the different functions of the same angle, were derived from the definitions [1], where the angle was assumed to be acute. But the same formulas can be derived from the general definitions [45], and thus shown to be true for angles of all magnitudes.

118. Thus, in particular, by tracing the cosecant $\frac{r}{y}$ and the sine $\frac{y}{r}$ through the four quadrants we see that these two functions are always the reciprocals of each other, and are positive or negative in the same quadrants.

The same is true of the cotangent $\frac{x}{y}$ and tangent $\frac{y}{x}$; also of the cosine $\frac{x}{r}$ and the secant $\frac{r}{x}$. Hence the formulas

$$\sin A \times \csc A = 1$$
$$\tan A \times \cot A = 1$$
$$\sec A \times \cos A = 1$$

are true for angles of all magnitudes.

119. Again, the tangent is always the quotient of the sine divided by the cosine. For if these functions be traced through the four quadrants, it will be seen that the tangent is positive in those quadrants where the sine and the cosine are alike, that is, both positive or both negative; and the tangent is negative where the sine and cosine are not alike. Similar reasoning applies to the cotangent. Hence the formulas

$$\tan A = \frac{\sin A}{\cos A} \qquad \cot A = \frac{\cos A}{\sin A}$$

are true for angles of all magnitudes.

120. And again, the squares of the coördinates of the point A being always positive, although the coördinates themselves may be positive or negative, Figs. 22, 23, 24, and 25, the equation

$$x^2+y^2=r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [46]$$

is true for all directions of the terminal line O A, and the three formulas

$$\sin^2 A+\cos^2 A=1,$$
$$\sec^2 A=1+\tan^2 A,$$
$$\csc^2 A=1+\cot^2 A,$$

which were derived from the equation $a^2+b^2=h^2$, may just as well be derived from $x^2+y^2=r^2$, and thus be shown to be true for angles of all magnitudes.

121. From the last three formulas are derived six quadratic solutions:

$$\sin A=\pm\sqrt{1-\cos^2 A} \quad . \quad . \quad . \quad . \quad . \quad . \quad [47]$$
$$\cos A=\pm\sqrt{1-\sin^2 A} \quad . \quad . \quad . \quad . \quad . \quad . \quad [48]$$
$$\tan A=\pm\sqrt{\sec^2 A-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [49]$$
$$\sec A=\pm\sqrt{1+\tan^2 A} \quad . \quad . \quad . \quad . \quad . \quad . \quad [50]$$
$$\cot A=\pm\sqrt{\csc^2 A-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad [51]$$
$$\csc A=\pm\sqrt{1+\cot^2 A} \quad . \quad . \quad . \quad . \quad . \quad . \quad [52]$$

which differ from the formulas [7], [8], [11], [12], [13], [14] only in having the double sign before the radical to indicate that they are *algebraic*, and not merely *arithmetical*, square roots.

122. The interpretation of these double solutions is found in the fact that to a given value of any one function belong two angles between 0° and 360°; and the other five functions of these two angles are arithmetically equal each to each; but four of them have opposite algebraic signs. These four are the ones which are derived from the given one by means of quadratic solutions. The fifth is simply the reciprocal of the given one, and therefore has the same value for the two angles.

To illustrate:

(i) Suppose that $\cos A = \frac{1}{2}$. Then A is either 60° or 300°. By [47], $\sin A = \pm\sqrt{1-(\frac{1}{2})^2} = \pm 0.866$; the first root $+0.866$ being the sine of 60°, and the second, -0.866 the sine of 300°. The secant being simply the reciprocal of the cosine has the same value, $+2$, for both 60° and 300°. By [49], $\tan A = \pm\sqrt{2^2-1} = \pm 1.732$; the first root, $+1.732$, being the tangent of 60°, and the second, $-1,732$, the tangent of 300°. By other formulas are found $\cot A = \pm 0.577$ and $\csc A = \pm 1.155$; the first root in each case being a function of 60° and the second a function of 300°.

These results may be found in their proper places in Table II.

(ii) Again, suppose that $\tan A = -1.192$. Then A is either 130° or 310°. By [50] $\sec A = \pm\sqrt{1+(-1.192)^2} = \pm 1.556$; the positive root being the secant of 310° and the negative root the secant of 130°. By [3], $\cos A = \frac{1}{\pm 1.556} = \pm 0.643$, the positive value being the cosine of 310° and the negative the cosine of 130°. By [47], $\sin A = \pm\sqrt{1-(\pm 0.643)^2} = \pm 0.766$; the positive root being the sine of 130° and the negative root the sine of 310°. By [3], $\csc A = \frac{1}{\pm 0.766} = \pm 1.305$; the positive value being the cosecant of 130° and the negative the cosecant of 310°. The cotangent being simply the reciprocal of the tangent has the same value, -0.839, for both 130° and 310°. These results may be found in Table II.

123. Formulas [3] to [14] having now been generalized, all formulas deduced from them may be regarded as true for angles of all magnitudes. This remark applies to the following table of formulas, in which each function is expressed in terms of each of the others. The derivation of this table from formulas already proved will afford a useful exercise.

For numerical illustrations of these formulas, refer to Table II. Taking any value of any function there given, compute, by aid of the formulas in this table, the other functions of the same angle, and see if the results agree with those there found.

FORMULAS EXPRESSING EACH FUNCTION IN TERMS OF EACH OF THE OTHERS.

$\sin A =$	$\tan A =$	$\sec A =$
$\sin A$	$\dfrac{\sin A}{\pm\sqrt{1-\sin^2 A}}$	$\dfrac{1}{\pm\sqrt{1-\sin^2 A}}$
$\dfrac{\tan A}{\pm\sqrt{1+\tan^2 A}}$	$\tan A$	$\pm\sqrt{1+\tan^2 A}$
$\dfrac{\pm\sqrt{\sec^2 A-1}}{\sec A}$	$\pm\sqrt{\sec^2 A-1}$	$\sec A$
$\pm\sqrt{1-\cos^2 A}$	$\dfrac{\pm\sqrt{1-\cos^2 A}}{\cos A}$	$\dfrac{1}{\cos A}$
$\dfrac{1}{\pm\sqrt{1+\cot^2 A}}$	$\dfrac{1}{\cot A}$	$\dfrac{\pm\sqrt{1+\cot^2 A}}{\cot A}$
$\dfrac{1}{\csc A}$	$\dfrac{1}{\pm\sqrt{\csc^2 A-1}}$	$\dfrac{\csc A}{\pm\sqrt{\csc^2 A-1}}$

$\cos A =$	**$\cot A =$**	**$\csc A =$**
$\pm\sqrt{1-\sin^2 A}$	$\dfrac{\pm\sqrt{1-\sin^2 A}}{\sin A}$	$\dfrac{1}{\sin A}$
$\dfrac{1}{\pm\sqrt{1+\tan^2 A}}$	$\dfrac{1}{\tan A}$	$\dfrac{\pm\sqrt{1+\tan^2 A}}{\tan A}$
$\dfrac{1}{\sec A}$	$\dfrac{1}{\pm\sqrt{\sec^2 A-1}}$	$\dfrac{\sec A}{\pm\sqrt{\sec^2 A-1}}$
$\cos A$	$\dfrac{\cos A}{\pm\sqrt{1-\cos^2 A}}$	$\dfrac{1}{\pm\sqrt{1-\cos^2 A}}$
$\dfrac{\cot A}{\pm\sqrt{1+\cot^2 A}}$	$\cot A$	$\pm\sqrt{1+\cot^2 A}$
$\dfrac{\pm\sqrt{\csc^2 A-1}}{\csc A}$	$\pm\sqrt{\csc^2 A-1}$	$\csc A$

THE FUNCTIONS REPRESENTED BY LINES.

124. By the earlier writers on the subject, the trigonometric functions were defined as *lines*, and they were considered to be functions, not of *angles*, but of *arcs.* But modern writers have generally preferred the simpler method of definition, which discards arcs and radii, and regards the functions as numerical ratios merely. It is often convenient, however, for the sake of clearness, to *represent* these functions by straight lines. These lines are so drawn in connection with a unit-circle (that is, a circle having a radius equal to a linear unit) that their lengths for a given arc (or angle) are numerically equal to the corresponding trigonometric ratios. What is valuable in the old definitions is retained in this method of representation.

125. The following is the method of representation: Draw four unit-circles, one for the angle in each quadrant, and divide each into quadrants, **O X** being the initial line of the first quadrant. (Figures 29, 30, 31, 32.)

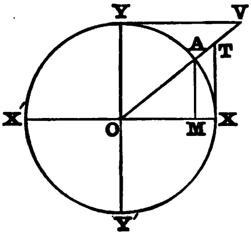

Fig. 29.
FUNCTIONS OF AN ANGLE IN THE FIRST QUADRANT.

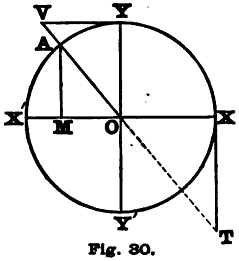

Fig. 30.
FUNCTIONS OF AN ANGLE IN THE SECOND QUADRANT.

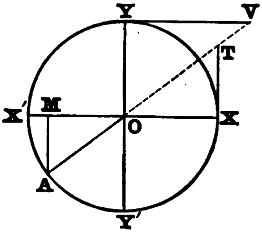

Fig. 31.
FUNCTIONS OF AN ANGLE IN THE THIRD QUADRANT.

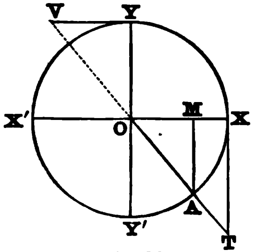

Fig. 32.
FUNCTIONS OF AN ANGLE IN THE FOURTH QUADRANT.

The angle whose functions are to be represented is **X O A**, measured by the arc **X A**. Denote this angle by A. The terminal radius, **O A**, is to be prolonged until it meets, in the points **T** and **V**, tangents to the circle drawn at the points **X** and **Y**. A perpendicular drawn from **A**, the end of the arc, upon the diameter passing through the beginning completes the construction.

Then, in each figure, since $OA = 1$, $OX = 1$, and $OY = 1$,

$$\sin A = \frac{\mathrm{MA}}{\mathrm{OA}} = \frac{\mathrm{MA}}{1} = \mathrm{MA}$$

$$\tan A = \frac{\mathrm{XT}}{\mathrm{OX}} = \frac{\mathrm{XT}}{1} = \mathrm{XT}$$

$$\sec A = \frac{\mathrm{OT}}{\mathrm{OX}} = \frac{\mathrm{OT}}{1} = \mathrm{OT}$$

$$\cos A = \frac{\mathrm{OM}}{\mathrm{OA}} = \frac{\mathrm{OM}}{1} = \mathrm{OM}$$

$$\cot A = \frac{\mathrm{YV}}{\mathrm{OY}} = \frac{\mathrm{YV}}{1} = \mathrm{YV}$$

$$\csc A = \frac{\mathrm{OV}}{\mathrm{OY}} = \frac{\mathrm{OV}}{1} = \mathrm{OV}$$

126. With this construction in mind, we may describe the functions of an arc or angle as follows:

The SINE **M A** is the perpendicular distance of the end of the arc from the diameter passing through the beginning.

The TANGENT **X T** is that part of a geometric tangent drawn at the beginning of the arc which is intercepted between the point of tangency and the diameter prolonged through the end of the arc.

The SECANT **O T** is the distance from the centre of the circle to the end of the tangent.

The COSINE **O M** is the distance from the centre of the circle to the foot of the sine.

The COTANGENT **Y V** is that part of a geometric tangent drawn at a quadrant's distance in advance of the beginning of the arc which is intercepted between the point of tangency and the diameter prolonged through the end of the arc.

The COSECANT **O V** is the distance from the centre of the circle to the end of the cotangent.

127. These lines represent the functions not only in numerical value but also in algebraic quality, as positive or negative.

Thus, when the angle is in the first or second quadrant its sine is positive, and the line **M A** has an upward direction; but when the angle is in the third or the fourth quadrant the sine is negative, and the line **M A** has a downward direction. As to the cosine, when the angle is in the first or fourth quadrant, making the cosine positive, the line **O M** extends to the right; but when the angle is in the second or third quadrant making the cosine negative, the line **O M** extends the opposite way.

As to the tangent and cotangent, when the angle is in the first or third quadrant making these functions positive, the line **X T** extends upward and **Y V** to the right; but when the angle is in the second or fourth quadrant making these functions negative, **X T** extends downward and **Y V** to the left.

As to the secant and the cosecant, the figures show that in those quadrants where the function is positive the representative line **O T** or **O V** has the *same* direction as **O A**, the terminal line of the angle; but in those quadrants where the function is negative the representative line has a direction *opposite* to that of **O A**. In the figures, negative secants and cosecants are represented by dotted lines.

128. The variations of the functions produced by the angle's increasing from 0° to 90°, from 90° to 180°, and so on, can be pictured to the mind, if we imagine the terminal line **O A** (prolonged), starting from the initial position **O X**, turning through the first quadrant to the position **O Y**, then through the second, third, and fourth quadrants back finally to its initial position. The lines representing the functions will grow longer or shorter as the terminal line turns through a quadrant; and when it passes from one quadrant into another, these lines will have the same limiting values as the functions have, and will change from positive to negative or from negative to positive in the same way. The columns of Table II may be regarded as records of the changes in the lengths and directions of these function-lines, as the terminal line of the angle passes through the quadrants.

129. It is worth observing, moreover, that the fundamental formulas [3] to [14] can be proved directly from the figures which represent the trigonometric functions as lines. The advantage of this is that the geometric figures aid the memory in retaining the formulas.

Thus in all the Figures, 29, 30, 31, and 32,

$$\overline{\mathrm{MA}}^2+\overline{\mathrm{OM}}^2=\overline{\mathrm{OA}}^2$$

But $\mathrm{OA}=1$, and by Art. 125

$$\mathrm{MA}=\sin A \qquad \mathrm{OM}=\cos A.$$

Therefore, by substitution,

$$\sin^2 A+\cos^2 A=1$$

which is [6].

The similar triangles **OMA** and **OXT** give the proportion

$$\mathrm{MA}:\mathrm{OM}=\mathrm{XT}:\mathrm{OX}$$

which, by substitution of values given in Art. 125, bebecomes

$$\sin A:\cos A=\tan A:1$$

or

$$\frac{\sin A}{\cos A}=\tan A$$

which is one equation of [4]

130. THE FUNCTIONS OF A NEGATIVE ANGLE.

Let A and A' be two angles of the same magnitude, but the former positive and the latter negative, so that

$$A' = -A$$

The functions of A' are to be found and compared with those of A. Four figures (Figs. 33, 34, 35, and 36) are

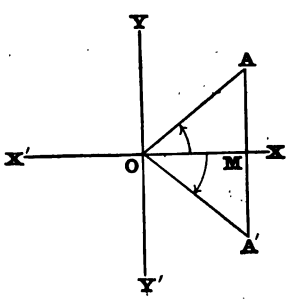

Fig. 33.

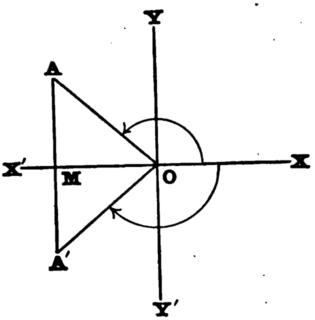

Fig. 34.

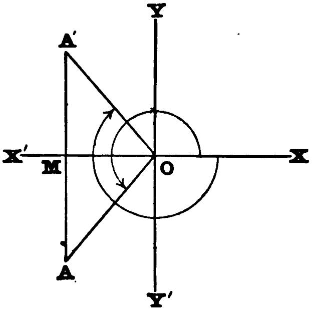

Fig. 35.

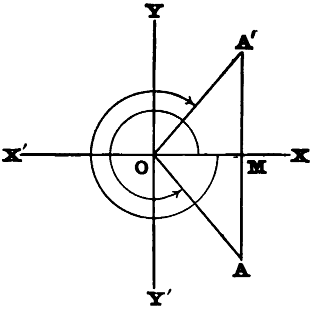

Fig. 36.

drawn to show the angles in each of the four quadrants, the line **O X** in each figure being the initial line and the lines **O A** and **O A′** the terminal lines of the angles A and A' respectively. The curves arrows in each figure show the amount and kind (positive or negative) of revolution necessary to generate the angles. In each figure the right triangles **O M A** and **O M A′** are equal; because **O A′** is taken of a length equal to **O A**, and the acute angles **M O A** and **M O A′** are, from their relations to A and A', geometrically equal. Hence the points **A** and **A′** have everywhere the same abscissa **O M**; but the ordinate of **A′** is always the negative of the ordinate of **A**; that is, denoting the coördinates of **A** by x and y and those of **A′** by x' and y',

$$x'=x \text{ and } y'=-y$$

Also, since terminal lines are always positive,

$$r'=r$$

Now

$$\sin A'=\frac{y'}{r'}=\frac{-y}{r}=-\frac{y}{r}=-\sin A$$

$$\tan A'=\frac{y'}{x'}=\frac{-y}{x}=-\frac{y}{x}=-\tan A$$

$$\sec A'=\frac{r'}{x'}=\frac{r}{x}=\sec A$$

$$\cos A'=\frac{x'}{r'}=\frac{x}{r}=\cos A$$

$$\cot A'=\frac{x'}{y'}=\frac{x}{-y}=-\frac{x}{y}=-\cot A$$

$$\csc A'=\frac{r'}{y'}=\frac{r}{-y}=-\frac{r}{y}=-\csc A$$

By writing in place of A' its equal, $-A$, these results become:

$$\left.\begin{array}{ll}\sin(-A)=-\sin A & \cos(-A)=\cos A\\ \tan(-A)=-\tan A & \cot(-A)=-\cot A\\ \sec(-A)=\sec A & \csc(-A)=-\csc A\end{array}\right\} \quad [53]$$

131. THE FUNCTIONS OF A + 90°.

The functions of the angle $A + 90°$ are thus found from those of the angle A. Draw four figures, one for the angle A in each quadrant (Figs. 37, 38, 39, and 40), and draw in each figure **O A′** ninety degrees in advance of **O A**. By taking **O A** and **O A′** of equal lengths, and drawing **M A** the ordinate of the point **A** and **M′ A′** the abscissa of the point **A′**, two right triangles **O M A** and **O M′ A′** are formed which are easily seen to be geometrically equal. Hence, in each figure, the abscissa of the

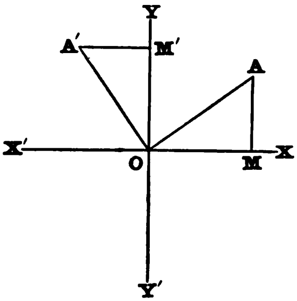

Fig. 37.

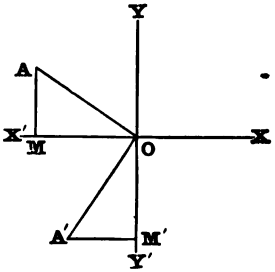

Fig. 38.

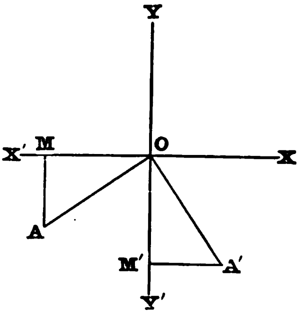

Fig. 39.

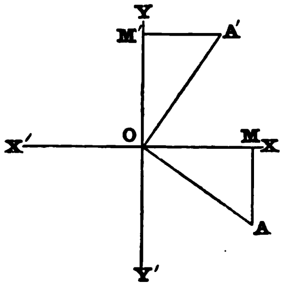

Fig. 40.

point **A**′ is equal in length to the ordinate of the point **A**; but whenever one is positive the other is negative; that is, in all cases

$$x' = -y$$

Also, the ordinate of the point **A**′ is equal in length to the abscissa of the point **A**; and whenever one is positive the other is positive; that is, in all cases

$$y' = x$$

Also, $$r' = r$$

Now, $\sin(A+90^\circ) = \frac{y'}{r'} = \frac{x}{r} = \cos A$

$$\cos(A+90^\circ) = \frac{x'}{r'} = \frac{-y}{r} = -\sin A,$$

and so on. The results are:

$$\left.\begin{array}{ll} \sin(A+90^\circ) = \cos A & \cos(A+90^\circ) = -\sin A \\ \tan(A+90^\circ) = -\cot A & \cot(A+90^\circ) = -\tan A \\ \sec(A+90^\circ) = -\csc A & \csc(A+90^\circ) = \sec A \end{array}\right\} . \quad [54]$$

These results, which are general formulas, may be thus illustrated by means of Table II.

(i) Suppose $A = 40^\circ$; then $A + 90^\circ = 130^\circ$; and, by the Table,

$$\begin{array}{ll} \sin 130^\circ = +0.766, & \tan 130^\circ = -1.192, \\ \cos 40^\circ = +0.766, & -\cot 40^\circ = -1.192, \text{ and so on.} \end{array}$$

(ii) Suppose $A = 200^\circ$; then $A + 90^\circ = 290^\circ$; and, by the Table,

$$\begin{array}{ll} \sin 290^\circ = -0.940, & \tan 290^\circ = -2.747, \\ \cos 200^\circ = -0.940, & -\cot 200^\circ = -2.747, \text{ and so on.} \end{array}$$

The formulas proved in the next two Articles can be illustrated in the same way.

132. THE FUNCTIONS OF A + 180°.

The functions of the angle $A + 180°$ are thus found from those of the angle A. Draw four figures, one for the angle A in each quadrant (Figs. 41, 42, 43, and 44), and draw in each figure **O A′** one hundred eighty degrees in advance of **O A**. Taking **O A** and **O A′** of equal lengths and drawing **A M** the ordinate of the point **A** and **A′ M′** the ordinate of the point **A′**, we have two right triangles which are easily seen to be geometrically equal. Hence the abscissa of the point **A′** is equal in length to

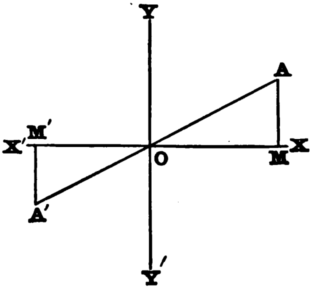

Fig. 41.

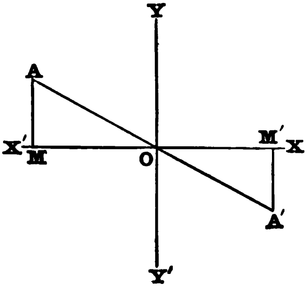

Fig. 42.

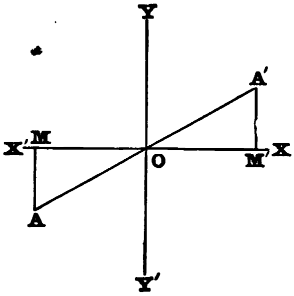

Fig. 43.

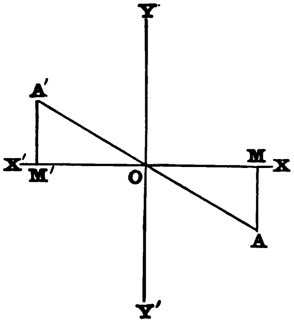

Fig. 44.

the abscissa of the point A, but whenever the one is positive the other is negative; and the same is true of the two ordinates; that is, in all cases,

$$x' = -x$$
$$y' = -y$$

Also,
$$r' = r$$

$$\text{Now, } \sin(A+180°) = \frac{y'}{r'} = \frac{-y}{r} = -\sin A$$

$$\cos(A+180°) = \frac{x'}{r'} = \frac{-x}{r} = -\cos A$$

and so on, with the following results:

$$\left.\begin{array}{ll} \sin(A+180°) = -\sin A & \cos(A+180°) = -\cos A \\ \tan(A+180°) = \tan A & \cot(A+180°) = \cot A \\ \sec(A+180°) = -\sec A & \csc(A+180°) = -\csc A \end{array}\right\} \quad [55]$$

133. THE FUNCTIONS OF A + 270°.

By drawing figures and reasoning as in the other cases it may be shown that

$$x' = y \text{ and } y' = -x,$$

and therefore

$$\left.\begin{array}{ll} \sin(A+270°) = -\cos A & \cos(A+270°) = \sin A \\ \tan(A+270°) = -\cot A & \cot(A+270°) = -\tan A \\ \sec(A+270°) = \csc A & \csc(A+270°) = -\sec A \end{array}\right\}. \quad [56]$$

THE FUNCTIONS OF A ± k 360°.

134. Since increasing or diminishing an angle A by 360° or by any number of times 360° brings the terminal line around to the same position it had after generating the angle A, the six functions of the angle $A \pm k\ 360°$

are identical with the six functions of the angle A, each with each. Thus,

$$\left.\begin{array}{l}\sin(A \pm k\,360^\circ) = \sin A \\ \cos(A \pm k\,360^\circ) = \cos A\end{array}\right\} \quad . \quad . \quad . \quad [57]$$

and so on

THE FUNCTIONS OF ANY ANGLE WHATEVER.

135. The use of the formulas [53] to [57] just proved in finding from Table I, or from any other table limited to the first quadrant, the functions of angles of all magnitudes positive and negative, may best be shown by a few examples.

(i) The natural functions of an angle in the second quadrant, as 125°.

If 125° be taken as equal to $A+90^\circ$, the value of A becomes 35°, and the formulas [**54**] give

$$\begin{array}{l}\sin 125^\circ = \sin(35^\circ + 90^\circ) = \cos 35^\circ \\ \cos 125^\circ = \cos(35^\circ + 90^\circ) = -\sin 35^\circ\end{array}$$

and so on.

But by Table I, $\cos 35^\circ = 0.819$,
$\sin 35^\circ = 0.574$,

and so on.

Hence,

$$\begin{array}{lll}\sin 125^\circ = +0.819, & \tan 125^\circ = -1.428, & \sec 125^\circ = -1.743, \\ \cos 125^\circ = -0.574, & \cot 125^\circ = -0.700, & \csc 125^\circ = +1.221.\end{array}$$

(ii) The natural functions of an angle in the third quadrant, as 246°.

If 246° be taken as equal to $A+180^\circ$, the value of A becomes 66°, and the formulas [55] give

$$\begin{aligned}\sin 246^\circ &= \sin(66^\circ + 180^\circ) = -\sin 66^\circ\\ \cos 246^\circ &= \cos(66^\circ + 180^\circ) = -\cos 66^\circ\end{aligned}$$

and so on.

Hence (Table I),

$$\begin{array}{lll}\sin 246^\circ = -0.914, & \tan 246^\circ = +2.246, & \sec 246^\circ = -2.459,\\ \cos 246^\circ = -0.407, & \cot 246^\circ = +0.445, & \csc 246^\circ = -1.095.\end{array}$$

(iii) The natural functions of an angle in the fourth quadrant, as 318°.

If 318° be taken as equal to $A + 270^\circ$, the value of A becomes 48°, and the formulas [56] give.

$$\begin{aligned}\sin 318^\circ &= \sin(48^\circ + 270^\circ) = -\cos 48^\circ\\ \cos 318^\circ &= \cos(48^\circ + 270^\circ) = \sin 48^\circ\end{aligned}$$

and so on.

Hence (Table I),

$$\begin{array}{lll}\sin 318^\circ = -0.669, & \tan 318^\circ = -0.900, & \sec 318^\circ = +1.346,\\ \cos 318^\circ = +0.743, & \cot 318^\circ = -1.111, & \csc 318^\circ = -1.494.\end{array}$$

(iv) The natural functions of an angle exceeding four right angles, as 3000°.

If 3000° be taken as equal to $A + k\ 360^\circ$, and the value of k (found by dividing 3000° by 360°) be taken as 8, we have

$$3000^\circ = A + 2880^\circ$$

so that the value of A becomes 120° and the formulas [57] give

$$\begin{aligned}\sin 3000^\circ &= \sin(120^\circ + 2880^\circ) = \sin 120^\circ\\ \cos 3000^\circ &= \cos(120^\circ + 2880^\circ) = \cos 120^\circ\end{aligned}$$

and so on, the functions of 3000° being identical in name and in value with the functions of 120°.

$$\begin{aligned}\cos 600° &= \cos(280° + 320°)\\ &= \cos 280° \cos 320° - \sin 280° \sin 320°\\ &= (0.174) \times (0.766) - (-0.985) \times (-0.643)\\ &= 0.133284 - 0.633355\\ &= -0.500\end{aligned}$$

These results agree with the values given in Table II for the sine and the cosine of $240° = 600° - 360°$.

THE SINE AND THE COSINE OF THE DIFFERENCE OF TWO ANGLES.

149. Let the angle **XOA** (Fig 54) be denoted by A, and the angle **AOB** by $-B$. Then the angle **XOB** is the algebraic sum of these, or $A-B$. The sine and the cosine of the angle $A-B$ are to be expressed in terms of the sines and the cosines of A and B.

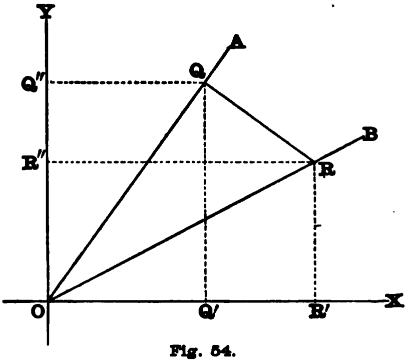

Fig. 54.

On **OB**, the terminal side of the angle $-B$, lay off **OR** of any chosen length. From the point **R** let fall a perpendicular **RQ** upon **OA**, the terminal sine of the angle A. Then, in the right triangle **OQR**, by [53],

$$\begin{aligned}QR &= OR \sin(-B) = -OR \sin B\\ OQ &= OR \cos(-B) = OR \cos B\end{aligned}$$

Now project the sides of this triangle upon the rectangular axes **OX** and **OY**. The projecting angle for the side **OQ** is A; that for the side **QR** is $A + 90°$, since it is the angle between **O X** and the positive direction of **Q R**, which is **R Q**; and that for the side **O R** is $A-B$.

The projections of **O Q** are

$$\text{O Q}' = \text{O Q} \cos A$$
$$\text{O Q}'' = \text{O Q} \sin A$$

or, by substituting for **O Q** its equivalent, **O R** cos B, as above given.

$$\text{O Q}' = \text{O R} \cos A \cos B$$
$$\text{O Q}'' = \text{O R} \sin A \cos B$$

The projections of **Q R** are

$$\text{Q}'\,\text{R}' = \text{Q R} \cos (A+90°) = -\text{Q R} \sin A$$
$$\text{Q}''\,\text{R}'' = \text{Q R} \sin (A+90°) = \text{Q R} \cos A$$

or, by substituting for **Q R** its equivalent, $-$ **O R** sin B, as above given.

$$\text{Q}'\,\text{R}' = \text{O R} \sin A \sin B$$
$$\text{Q}''\,\text{R}'' = -\text{O R} \cos A \sin B$$

The projections of **O R** are

$$\text{O R}' = \text{O R} \cos (A-B)$$
$$\text{O R}'' = \text{O R} \sin (A-B)$$

The projection of one side of a triangle being equal to the sum of the projections of the other two sides, we have [58] the two equations

$$\text{O R}'' = \text{O Q}'' + \text{Q}''\,\text{R}''$$
$$\text{O R}' = \text{O Q}' + \text{Q}'\,\text{R}'$$

which, by substituting the equivalents above found and dividing through by **O R**, become

$$\sin (A-B) = \sin A \cos B - \cos A \sin B \quad . \quad . \quad . \quad [63]$$
$$\cos (A-B) = \cos A \cos B + \sin A \sin B \quad . \quad . \quad . \quad [64]$$

That this proof is perfectly general may be shown by drawing the figure in different forms and repeating the proof with each form, as was suggested in Article 147.

150. The striking similarity between the proof just given and that given for the formulas [61] and [62] arises from the fact that the two proofs are, in so far as the reasoning is concerned, identical, the only differences being certain differences of expression occasioned by the use of the negative angle $-B$ instead of the positive angle $+B$. Indeed, formulas [63] and [64] are simply what [61] and [62] become by the substitution of $-B$ for $+B$ therein.

151. This fact suggests that the geometrical proof of [63] and [64] may be set aside as unnecessary, and that these formulas may be derived from [61] and [62] by a purely analytic process. The following is such a process.

Assuming B to be less than A, we put

$$A-B=C$$

and, consequently,

$$A=B+C$$

Formulas [61] and [62] applied to the sum of the two angles, B and C, give

$$\sin(B+C)=\sin B\cos C+\cos B\sin C$$
$$\cos(B+C)=\cos B\cos C-\sin B\sin C$$

which, by writing A in place of its equivalent $B+C$, and $A-B$ in place of its equivalent C, become

$$\sin A=\sin B\cos(A-B)+\cos B\sin(A-B)$$
$$\cos A=\cos B\cos(A-B)-\sin B\sin(A-B)$$

By eliminating $\cos(A-B)$ from these two equations we get [63]; and by eliminating $\sin(A-B)$ we get [64].

Thus:

(i) If the first equation be multiplied through by $\cos B$ and the second by $\sin B$, the result of subtracting the second from the first is

$$\sin A \cos B - \cos A \sin B = (\cos^2 B + \sin^2 B) \sin (A - B);$$

or, since $\cos^2 B + \sin^2 B = 1$,

$$\sin A \cos B - \cos A \sin B = \sin (A - B),$$

which is [63].

(ii) If the first equation be multiplied through by $\sin B$ and the second by $\cos B$, the result of adding the two equations is

$$\sin A \sin B + \cos A \cos B = (\sin^2 B + \cos^2 B) \cos (A - B)$$

or, since $\sin^2 B + \cos^2 B = 1$,

$$\sin A \sin B + \cos A \cos B = \cos (A - B)$$

which is [64].

THE TANGENT AND THE COTANGENT OF THE SUM AND OF THE DIFFERENCE OF TWO ANGLES.

152. By [4], [61], and [62],

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

or, dividing both numerator and denominator through by $\cos A \cos B$, and reducing each term by [4],

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad . \; . \; . \; . \; . \; . \; . \; . \quad [65]$$

In the same way, from [63] and [64] is derived

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad . \; . \; . \; . \; . \; . \; . \; . \quad [66]$$

By [62], [61], and [4],

$$\cot (A + B) = \frac{\cos (A + B)}{\sin (A + B)} = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$$

or, dividing both numerator and denominator through by sin A sin B, and reducing each term by [4],

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A} \quad . \; . \; . \; . \; . \; . \; . \; . \quad [67]$$

In the same way, from [63] and [64] is derived

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A} \quad . \; . \; . \; . \; . \; . \; . \; . \quad [68]$$

FUNCTIONS OF THE DOUBLE AND OF THE HALF OF AN ANGLE.

153. In [61], [62], [65], and [67], let $B=A$. Then

$$\sin 2A=2\sin A\cos A \quad . \; . \; . \; . \; . \; . \; . \; . \quad [69]$$

$$\cos 2A=\cos^2 A-\sin^2 A \quad . \; . \; . \; . \; . \; . \; . \; . \quad [70]$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [71]$$

$$\cot 2A=\frac{\cot^2 A-1}{2\cot A} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [72]$$

If the members of [69] be added to and subtracted from the members of

$$1=\cos^2 A+\sin^2 A$$

the results will be

$$1+\sin 2A=(\sin A+\cos A)^2 \quad . \; . \; . \; . \; . \; . \; . \quad [73]$$

$$1-\sin 2A=(\sin A-\cos A)^2 \quad . \; . \; . \; . \; . \; . \; . \quad [74]$$

And, if the members of [70] be treated in the same way, the results will be

$$1+\cos 2A=2\cos^2 A \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [75]$$

$$1-\cos 2A=2\sin^2 A \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [76]$$

If, in the last eight formulas, we write A in place of $2A$, and accordingly $\frac{1}{2}A$ in place of A, we have

$$\sin A=2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A \quad . \; . \; . \; . \; . \; . \; . \; . \quad [77]$$

$$\cos A = \cos^2 \tfrac{1}{2} A - \sin^2 \tfrac{1}{2} A. \qquad [78]$$

$$\tan A = \frac{2 \tan \frac{1}{2} A}{1 - \tan^2 \frac{1}{2} A} \qquad [79]$$

$$\cot A = \frac{\cot^2 \frac{1}{2} A - 1}{2 \cot \frac{1}{2} A} \qquad [80]$$

$$1 + \sin A = (\sin \tfrac{1}{2} A + \cos \tfrac{1}{2} A)^2 \qquad [81]$$

$$1 - \sin A = (\sin \tfrac{1}{2} A - \cos \tfrac{1}{2} A)^2 \qquad [82]$$

$$1 + \cos A = 2 \cos^2 \tfrac{1}{2} A \qquad [83]$$

$$1 - \cos A = 2 \sin^2 \tfrac{1}{2} A \qquad [84]$$

The last two equations give

$$\sin \tfrac{1}{2} A = \pm \sqrt{\tfrac{1}{2}(1 - \cos A)} \qquad [85]$$

$$\cos \tfrac{1}{2} A = \pm \sqrt{\tfrac{1}{2}(1 + \cos A)} \qquad [86]$$

$$\tan \tfrac{1}{2} A = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \qquad [87]$$

$$\cot \tfrac{1}{2} A = \pm \sqrt{\frac{1 + \cos A}{1 - \cos A}} \qquad [88]$$

If numerator and denominator of the fractions under the radical signs in the last two formulas be multiplied by $1 + \cos A$ or by $1 - \cos A$, and $\sin^2 A$ be substituted for $1 - \cos^2 A$ wherever the latter quantity occurs, the fractions will become perfect squares, and by taking the roots we shall have

$$\tan \tfrac{1}{2} A = \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A} \qquad [90]$$

$$\cot \tfrac{1}{2} A = \frac{1 + \cos A}{\sin A} = \frac{\sin A}{1 - \cos A} \qquad [91]$$

The double sign ± is not necessary in these two formulas; for the quantities $1 + \cos A$ and $1 - \cos A$ are always positive, while $\sin A$ and $\tan \frac{1}{2} A$ are always positive or negative simultaneously. The reason for this is that when A is an angle in the first or second quadrant so that sin A is positive, $\frac{1}{2}A$ is an angle in the first or third quadrant so that tan $\frac{1}{2}A$ is also positive; and when A is in the third or fourth quadrant, so that sin A is negative, $\frac{1}{2}A$ is in the second or fourth quadrant so that tan $\frac{1}{2}A$ is negative also.

SUMS AND DIFFERENCES OF FUNCTIONS.

154. The formulas [61], [62], [63], and [64] applied to the sum and difference of the two angles U and V give

$$\sin(U+V)=\sin U\cos V+\cos U\sin V$$
$$\sin(U-V)=\sin U\cos V-\cos U\sin V$$

$$\cos(U+V)=\cos U\cos V-\sin U\sin V$$
$$\cos(U-V)=\cos U\cos V+\sin U\sin V$$

whence, by addition and subtraction,

$$\sin(U+V)+\sin(U-V)=2\sin U\cos V$$
$$\sin(U+V)-\sin(U-V)=2\cos U\sin V$$
$$\cos(U+V)+\cos(U-V)=2\cos U\cos V$$
$$-\cos(U+V)+\cos(U-V)=2\sin U\sin V$$

Now, put

$$U+V=A$$
$$U-V=B$$

and, consequently

$$U=\tfrac{1}{2}(A+B)$$
$$V=\tfrac{1}{2}(A-B)$$

and substitute these equivalents in the foregoing equations. The result is

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad . \quad . \quad [92]$$
$$\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad . \quad . \quad [93]$$
$$\cos A+\cos B=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad . \quad . \quad [94]$$
$$-\cos A+\cos B=2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad . \quad . \quad [95]$$

If in these four equations we put $B=0$, we shall obtain formulas [77], [83], and [84], already proved.

If [92] be divided by [93], the result will be

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\tan \tfrac{1}{2}(A+B)\cot \tfrac{1}{2}(A-B)$$

or

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \quad \ldots\ldots\ldots \quad [96]$$

In the same way, from [94] and [95], we have

$$\frac{\cos A+\cos B}{-\cos A+\cos B}=\cot \tfrac{1}{2}(A+B)\cot \tfrac{1}{2}(A-B) \quad \ldots \quad [97]$$

and from [92], [93], and [94],

$$\frac{\sin A\pm\sin B}{\cos A+\cos B}=\tan \tfrac{1}{2}(A\pm B) \quad \ldots\ldots\ldots \quad [98]$$

and from [92], [93], and [95],

$$\frac{\sin A\pm\sin B}{-\cos A+\cos B}=\cot \tfrac{1}{2}(A\mp B) \quad \ldots\ldots\ldots \quad [99]$$

155. The product of [61] multiplied by [63] member by member is

$$\sin(A+B)\sin(A-B)=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$$

the second member of which becomes, by the substitution of $1-\sin^2 B$ for $\cos^2 B$, and reducing,

$$\sin^2 A-\sin^2 B$$

or, by the substitution of $1-\cos^2 A$ for $\sin^2 A$, and reducing,

$$\cos^2 B-\cos^2 A.$$

Hence

$$\left.\begin{aligned}\sin(A+B)\sin(A-B) &= \sin^2 A-\sin^2 B\\ &= (\sin A+\sin B)(\sin A-\sin B)\\ &= \cos^2 B-\cos^2 A\\ &= (\cos B+\cos A)(\cos B-\cos A)\end{aligned}\right\}[100]$$

And in the same way, from [62] and [64], may be deduced

$$\left.\begin{aligned}\cos(A+B)\cos(A-B) &= \cos^2 A-\sin^2 B\\ &= (\cos A+\sin B)(\cos A-\sin B)\\ &= \cos^2 B-\sin^2 A\\ &= (\cos B+\sin A)(\cos B-\sin A)\end{aligned}\right\}[101]$$

PARTICULAR RESULTS DEDUCED FROM THE GENERAL FORMULAS.

156. The general formulas of this chapter contain many particular formulas and results, some of which have aleady been found in other ways, but all of which are easily obtained from the general formulas by making particular suppositions as to the values of A and B. The following exercises lead to the more important of these particular results.

157. At the outset we shall suppose the numerical values of the functions of all angles to be unknown. The only facts assumed are the following,

$$\begin{aligned}\cos 45^\circ &= \sin 45^\circ\\ \sin 60^\circ &= \cos 30^\circ\\ \cos 60^\circ &= \sin 30^\circ\end{aligned}$$

which are geometrically evident; and the formula

$$\sin^2 A+\cos^2 A=1.$$

158. To find the functions of 0°.

In [63] and [64], make $B=A$. Then

$$\sin(A-A)=\sin A\cos A-\cos A\sin A=0$$
$$\cos(A-A)=\cos A\cos A+\sin A\sin A$$
$$=\cos^2 A+\sin^2 A=1$$

Therefore

$$\sin 0^\circ=0$$
$$\cos 0^\circ=1$$
$$\tan 0^\circ=\frac{\sin 0^\circ}{\cos 0^\circ}=\frac{0}{1}=0$$
$$\cot 0^\circ=\frac{\cos 0^\circ}{\sin 0^\circ}=\frac{1}{0}=\infty$$
$$\sec 0^\circ=\frac{1}{\cos 0^\circ}=\frac{1}{1}=1$$
$$\csc 0^\circ=\frac{1}{\sin 0^\circ}=\frac{1}{0}=\infty$$

159. To find the functions of 45°.

In [64], make $B=A=45^\circ$. Then

$$\cos(45^\circ-45^\circ)=\cos 45^\circ\cos 45^\circ+\sin 45^\circ\sin 45^\circ$$
$$\cos 0^\circ=\cos^2 45^\circ+\sin^2 45^\circ$$

or, since $\cos 0^\circ=1$ and $\cos 45^\circ=\sin 45^\circ$,

$$1=2\sin^2 45^\circ$$

whence

$$\sin^2 45^\circ=\tfrac{1}{2}$$
$$\sin 45^\circ=\sqrt{\tfrac{1}{2}}$$

Also

$$\cos 45^\circ=\sqrt{\tfrac{1}{2}}$$
$$\tan 45^\circ=\frac{\sin 45^\circ}{\cos 45^\circ}=\frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}}=1$$

and so on.

160. To find the functions of 30° and of 60°.

In [61] and [62], make $B=A=30°$. Then

$$\sin(30°+30°)=\sin 30° \cos 30° + \cos 30° \sin 30°$$
$$\sin 60° = 2 \sin 30° \cos 30°$$

or, since $\sin 60° = \cos 30°$,

$$\cos 30° = 2 \sin 30° \cos 30°$$

whence, by dividing each member by cos 30°,

$$1 = 2 \sin 30°$$
$$\sin 30° = \tfrac{1}{2}$$

Also

$$\cos(30°+30°) = \cos 30° \cos 30° - \sin 30° \sin 30°$$
$$\cos 60° = \cos^2 30° - \sin^2 30°$$

or, since $\cos 60° = \sin 30°$ and $\sin 30° = \tfrac{1}{2}$,

$$\tfrac{1}{2} = \cos^2 30° - \tfrac{1}{4}$$
$$\cos^2 30° = \tfrac{3}{4}$$
$$\cos 30° = \tfrac{1}{2}\sqrt{3} = \sin 60°$$
$$\tan 30° = \frac{\tfrac{1}{2}}{\tfrac{1}{2}\sqrt{3}} = \sqrt{\tfrac{1}{3}} = \cot 60°$$

and so on.

161. To find the functions of 90°.

In [61] and [62], put $A=45°$ and $B=45°$. Then

$$\sin 90° = \sin(45°+45°) = \sin 45° \cos 45° + \cos 45° \sin 45°$$
$$= (\sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{2}}) + (\sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{2}}) = \tfrac{1}{2} + \tfrac{1}{2} = 1.$$
$$\cos 90° = \cos(45°+45°) = \cos 45° \cos 45° - \sin 45° \sin 45°$$
$$= (\sqrt{\tfrac{1}{2}}\ \sqrt{\tfrac{1}{2}}) - (\sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{2}}) = \tfrac{1}{2} - \tfrac{1}{2} = 0$$

Or, put $A=60°$ and $B=30°$. Then

$$\sin 90° = \sin(60°+30°) = \sin 60° \cos 30° + \cos 60° \sin 30°$$
$$= (\tfrac{1}{2}\sqrt{3} \times \tfrac{1}{2}\sqrt{3}) + (\tfrac{1}{2} \times \tfrac{1}{2}) = \tfrac{3}{4} + \tfrac{1}{4} = 1$$

$$\begin{aligned}\cos 90° = \cos(60° + 30°) &= \cos 60° \cos 30° - \sin 60° \sin 30° \\ &= (\tfrac{1}{2} \times \tfrac{1}{2}\sqrt{3}) - (\tfrac{1}{2}\sqrt{3} \times \tfrac{1}{2}) \\ &= \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} = 0\end{aligned}$$

162. To find the functions of 180°.

In [61] and [62], make $A = 90°$ and $B = 90°$. Then

$$\begin{aligned}\sin 180° = \sin(90° + 90°) &= \sin 90° \cos 90° + \cos 90° \sin 90° \\ &= (1 \times 0) + (0 \times 1) = 0 \\ \cos 180° = \cos(90° + 90°) &= \cos 90° \cos 90° - \sin 90° \sin 90° \\ &= (0 \times 0) - (1 \times 1) = -1\end{aligned}$$

163. To find the functions of 270°.

In [61] and [62], make $A = 180°$ and $B = 90°$. Then

$$\begin{aligned}\sin 270° = \sin(180° + 90°) &= \sin 180° \cos 90° + \cos 180° \sin 90° \\ &= (0 \times 0) + (-1 \times 1) = -1 \\ \cos 270° = \cos(180° + 90°) &= \cos 180° \cos 90° - \sin 180° \sin 90° \\ &= (-1 \times 0) - (0 \times 1) = 0\end{aligned}$$

164. To find the functions of 360°.

In [61] and [62], make $A = 180°$ and $B = 180°$. Then

$$\begin{aligned}\sin 360° &= \sin(180° + 180°) \\ &= \sin 180° \cos 180° + \cos 180° \sin 180° \\ &= (0 \times -1) + (-1 \times 0) = 0 \\ \cos 360° &= \cos(180° + 180°) \\ &= \cos 180° \cos 180° - \sin 180° \sin 180° \\ &= (-1 \times -1) - (0 \times 0) = 1\end{aligned}$$

Or, make $A = 270°$ and $B = 90°$.

165. To find the functions of 450°, 540°, 630°, 720°, 1080°, and so on, substitute results already found in the formulas [61] and [62].

166. To find the functions of 120°.

$$\begin{aligned}\sin 120° &= \sin (60°+60°) = 2 \sin 60° \cos 60° \\ &= 2\times\tfrac{1}{2}\sqrt{3}\times\tfrac{1}{2} = \tfrac{1}{2}\sqrt{3} \\ \cos 120° &= \cos (60°+60°) = \cos^2 60° - \sin^2 60° \\ &= (\tfrac{1}{2})^2 - (\tfrac{1}{2}\sqrt{3})^2 = -\tfrac{1}{2}\end{aligned}$$

Or,

$$\begin{aligned}\sin 120° &= \sin (90°+30°) = \sin 90° \cos 30° + \cos 90° \sin 30° \\ &= (1\times\tfrac{1}{2}\sqrt{3}) + (0\times\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{3} \\ \cos 120° &= \cos (90°+30°) = \cos 90° \cos 30° - \sin 90° \sin 30° \\ &= (0\times\tfrac{1}{2}\sqrt{3}) - (1\times\tfrac{1}{2}) = -\tfrac{1}{2}\end{aligned}$$

Or again, by [63] and [64],

$$\begin{aligned}\sin 120° &= \sin (180°-60°) \\ &= \sin 180° \cos 60° - \cos 180° \sin 60° \\ &= (0\times\tfrac{1}{2}) - (-1\times\tfrac{1}{2}\sqrt{3}) = \tfrac{1}{2}\sqrt{3} \\ \cos 120° &= \cos (180°-60°) \\ &= \cos 180° \cos 60° + \sin 180° \sin 60° \\ &= (-1\times\tfrac{1}{2}) + (0\times\tfrac{1}{2}\sqrt{3}) = -\tfrac{1}{2}\end{aligned}$$

All these ways make the sine positive and the cosine negative, as they should be, for the second quadrant.

167. By substitution of previously ascertained values in the formulas, find the functions of 135°, 150°, 210°, 225°, 240°, 300°, 315°, 330°, 390°, 420°, 495°, 600°.

168. To find the functions of a negative angle.

In [63] and [64], make $A=0°$. Then

$$\begin{aligned}\sin (-B) &= \sin 0° \cos B - \cos 0° \sin B \\ &= (0\times\cos B) - (1\times\sin B) = -\sin B \\ \cos (-B) &= \cos 0° \cos B + \sin 0° \sin B \\ &= (1\times\cos B) + (0\times\sin B) = \cos B\end{aligned}$$

Compare these results with [53].

169. By giving proper values to B or to A in formulas [61] and [62] or in [63] and [64], the following special formulas are found, some of which have already been proved, but all are brought together here for convenient reference. (See Arts. 131–134.)

$$\left.\begin{aligned}\sin(A+90^\circ)&=\cos A\\ \cos(A+90^\circ)&=-\sin A\end{aligned}\right\}\quad\ldots\ldots\quad[102]$$

$$\left.\begin{aligned}\sin(A+180^\circ)&=-\sin A\\ \cos(A+180^\circ)&=-\cos A\end{aligned}\right\}\quad\ldots\ldots\quad[103]$$

$$\left.\begin{aligned}\sin(A+270^\circ)&=-\cos A\\ \cos(A+270^\circ)&=\sin A\end{aligned}\right\}\quad\ldots\ldots\quad[104]$$

$$\left.\begin{aligned}\sin(A+360^\circ)&=\sin A\\ \cos(A+360^\circ)&=\cos A\end{aligned}\right\}\quad\ldots\ldots\quad[105]$$

$$\left.\begin{aligned}\sin(A-90^\circ)&=-\cos A\\ \cos(A-90^\circ)&=\sin A\end{aligned}\right\}\quad\ldots\ldots\quad[106]$$

$$\left.\begin{aligned}\sin(A-180^\circ)&=-\sin A\\ \cos(A-180^\circ)&=-\cos A\end{aligned}\right\}\quad\ldots\ldots\quad[107]$$

$$\left.\begin{aligned}\sin(A-270^\circ)&=\cos A\\ \cos(A-270^\circ)&=-\sin A\end{aligned}\right\}\quad\ldots\ldots\quad[108]$$

$$\left.\begin{aligned}\sin(A-360^\circ)&=\sin A\\ \cos(A-360^\circ)&=\cos A\end{aligned}\right\}\quad\ldots\ldots\quad[109]$$

$$\left.\begin{aligned}\sin(90^\circ-B)&=\cos B\\ \cos(90^\circ-B)&=\sin B\end{aligned}\right\}\quad\ldots\ldots\quad[110]$$

$$\left.\begin{aligned}\sin(180^\circ-B)&=\sin B\\ \cos(180^\circ-B)&=-\cos B\end{aligned}\right\}\quad\ldots\ldots\quad[111]$$

$$\left.\begin{aligned}\sin(270^\circ-B)&=-\cos B\\ \cos(270^\circ-B)&=-\sin B\end{aligned}\right\}\quad\ldots\ldots\quad[112]$$

$$\left.\begin{aligned}\sin(360^\circ-B)&=-\sin B\\ \cos(360^\circ-B)&=\cos B\end{aligned}\right\}\quad\ldots\ldots\quad[113]$$

CHAPTER VII.

PLANE TRIANGLES.

170. In this chapter will be proved some general formulas relating to plane triangles, and from them will be deduced certain methods of solving triangles not given in the earlier chapters.

171. *Each side of a triangle is equal to the algebraic sum of the projections upon it of the other two sides.*

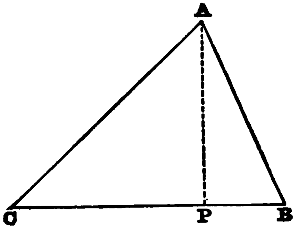

Fig. 55.

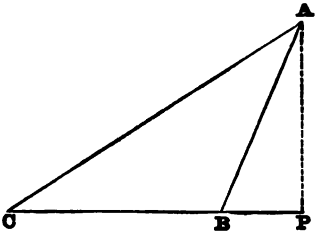

Fig. 56.

Let a perpendicular be drawn from any vertex, as A, to the opposite side (Fig. 55), or to the opposite side prolonged (Fig. 56). In both figures, by [43],

$$a = \text{CB} = \text{CP} + \text{PB}.$$

But

$$\text{CP} = \text{projection of CA} = b \cos C$$
$$\text{PB} = \text{projection of AB} = c \cos B$$

Therefore, by substitution,

$$a = b \cos C + c \cos B.$$

It is to be observed here, that, when the perpendicular falls within the triangle (Fig. 55) both projecting angles C and B must be acute, and both projections are positive, so that the sum **CP + PB** is an arithmetical sum; but, when the perpendicular falls without the triangle (Fig. 56) one of the projecting angles, as B, must be obtuse, and the corresponding projection is negative, so that the sum **CP + PB** is an arithmetical difference. The formula embraces both these cases, because the second member is an algebraic sum, either term of which may become negative in consequence of the angle's being in the second quadrant.

172. If the perpendicular had been drawn from **B**, the formula proved would have been

$$b = c \cos A + a \cos C,$$

and if from **C**

$$c = a \cos B + b \cos A.$$

A full expression of the theorem above stated is given by a set of three equations:

$$\left.\begin{aligned} a &= b \cos C + c \cos B \\ b &= c \cos A + a \cos C \\ c &= a \cos B + b \cos A \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad [114]$$

173. It will be observed, as we go on, that formulas relating to triangles usually occur in sets of three. Each equation of the set is just like the others in form, and differs only in the letters denoting sides and angles. When one equation of a set has been proved, the other two can be inferred from it by merely changing the letters. This change in the letters is simply a consequence of beginning the construction of the figure with another side or another angle than the one first chosen. The reasoning remains exactly the same.

In deriving one formula from another by changing the letters, there is a fixed order which should always be observed. It is this: Change A to B, B to C, and C to A;

also a to b, b to c, and c to a. The change is cyclic, and the process may be described as *a cyclic change of letters.* This process applied to the first equation of the set just proved [114] gives the second, applied to the second it gives the third, and applied to the third it gives the first again, thus completing the cycle.

174. The equations just proved may be taken as the fundamental equations of plane triangles; that is, all other general formulas expressing the properties of plane triangles can be deduced from these by purely algebraic transformations. On the other hand, all the general formulas can be proved independently of one another by geometric reasoning, so that one formula has no more claim than another to be regarded as fundamental.

Accordingly, in the further treatment of our subject, we have a choice of methods. By the analytic method, we can deduce all results from the fundamental equations [114], without further reference to the geometric relations of the parts; by the synthetic method we can establish each result separately, reasoning directly from the geometric relations of the parts. In the present chapter, we shall use both methods of treatment for all the principal formulas; the analytic, to show how all the formulas are derived from the fundamental one; and the synthetic, to keep the geometric signification of each formula clearly before the mind.

175. *The sides of a plane triangle are proportional to the sines of the opposite angles.*

ANALYTIC PROOF. The elimination of either a, b, or c from [114] will give an equation expressing this theorem. Thus, by substituting for c in the first equation its equivalent, the second member of the third equation, we have

$$a=b\cos C+a\cos^2 B+b\cos A\cos B$$

or, by factoring,

$$a(1-\cos^2 B)=b(\cos C+\cos A\cos B)$$

or, by substituting $\sin^2 B$ for $1-\cos^2 B$,

$$a\sin^2 B=b(\cos C+\cos A\cos B)$$

But

$$\begin{aligned} C&=180^\circ-(A+B)\\ \cos C&=\cos[180^\circ-(A+B)]\\ &=-\cos(A+B)\\ &=-\cos A\cos B+\sin A\sin B\end{aligned}$$

so that the last equation above becomes, by substitution in the second member,

$$a\sin^2 B=b\sin A\sin B$$

whence, by division,

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

Eliminating in the same way a from the second equation of [114], or b from the third, gives the result,

$$\frac{b}{\sin B}=\frac{c}{\sin C}\quad\text{or}\quad\frac{c}{\sin C}=\frac{a}{\sin A}.$$

All three results are embraced in the formula,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\quad\ldots\ldots\quad[115]$$

The common value of the three equal ratios [115] is called *the modulus of the triangle*, and, for a reason which will appear later, is denoted by $2R$.

$$2R=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\quad\ldots\quad[116]$$

176. *The sides of a plane triangle are proportional to the sines of the opposite angles.*

GEOMETRIC PROOF. Drawing from any vertex, as C, a perpendicular to the opposite side (Fig. 57), or to the opposite side prolonged (Fig. 58), we have, in the right triangles so formed,

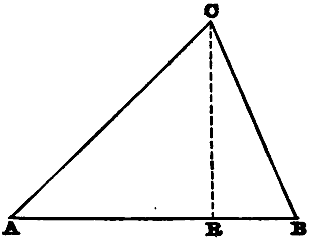

Fig. 57.

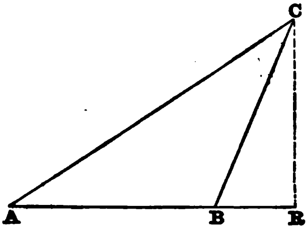

Fig. 58.

Fig. 57.	Fig. 58.
(i) $\text{RC}=\text{AC}\sin A$	(i) $\text{RC}=\text{AC}\sin A$
$= b\sin A$	$= b\sin A$
(ii) $\text{RC}=\text{BC}\sin B$	(ii) $\text{RC}=\text{BC}\sin \text{RBC}$
$= a\sin B$	$=a\sin(180^\circ-B)$
	$=a\sin B$

Therefore, by comparing (i) and (ii) in both cases,

$$a\sin B=b\sin A$$

whence

$$\frac{a}{\sin A}=\frac{b}{\sin B}.$$

By drawing the perpendicular from A or from B, the result would be

$$\frac{b}{\sin B}=\frac{c}{\sin C} \quad \text{or} \quad \frac{c}{\sin C}=\frac{a}{\sin A}.$$

Thus we have proved geometrically the three equations in [115].

177. *In a plane triangle, the sine of any angle is equal to the quotient of the opposite side divided by the diameter of the circumscribed circle.*

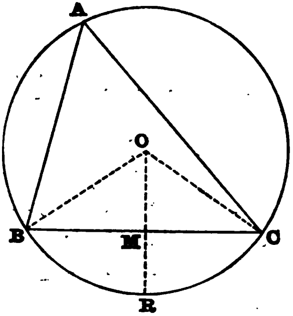

Fig. 59.

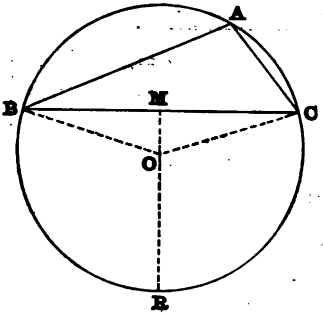

Fig. 60.

GEOMETRIC PROOF. From the centre of the circumscribed circle draw a perpendicular to the side **BC**. This bisects the side at **M**, and the arc **BC** at **R**. Draw radii **OB** and **OC**.

The angle A, being measured by one half of the arc **BC**, is equal to the angle **BOR**, which is also measured by one half of the arc **BC**. Therefore

$$\sin A = \sin \mathrm{BOR} = \frac{\mathrm{MB}}{\mathrm{OB}}.$$

But $\mathrm{MB} = \frac{1}{2}\,\mathrm{BC} = \frac{1}{2}\,a$; and **OB**, the radius of the circumscribed circle, may be denoted by R. Therefore, by substitution,

$$\sin A = \frac{\frac{1}{2}\,a}{R} = \frac{a}{2\,R}$$

which was to be proved.

In the same way may be proved

$$\sin B = \frac{b}{2\,R} \quad \text{and} \quad \sin C = \frac{c}{2\,R}$$

178. A comparison of the equations just proved with [116] shows that *the diameter of the circumscribed circle is numerically equal to the modulus of the triangle.*

A convenient form of stating the relations just proved is the following:

$$\left.\begin{aligned} a &= 2\,R \sin A \\ b &= 2\,R \sin B \\ c &= 2\,R \sin C \end{aligned}\right\} \quad \ldots\ldots\ldots \quad [117]$$

179. *The sum of two sides of a plane triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

ANALYTIC PROOF. The first two equations of [117] given by addition and subtraction,

$$a+b=2\,R\,(\sin A+\sin B)$$
$$a-b=2\,R\,(\sin A-\sin B)$$

whence, by division,

$$\frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}$$

But, by [96], the second member of this equation is equal to

$$\frac{\tan \frac{1}{2}\,(A+B)}{\tan \frac{1}{2}\,(A-B)}$$

Therefore

$$\frac{a+b}{a-b}=\frac{\tan \frac{1}{2}\,(A+B)}{\tan \frac{1}{2}\,(A-B)} \quad \ldots\ldots \quad [118]^*$$

Two other equations can be proved in the same way, or they may be deduced from this one by a cyclic change of letters. These equations express the theorem.

---

* NOTE. If $b>a$, and therefore $B>A$ both denominators in [118], are negative, but the equation holds true. If desirable to clear it of negative factors, it may be written $\frac{b+a}{b-a}=\frac{\tan \frac{1}{2}\,(B+A)}{\tan \frac{1}{2}\,(B-A)}$.

180. GEOMETRIC PROOF. With C as a centre, and with CA, the shorter of the two sides meeting at C, as a radius, draw a circle cutting the side CB in D. Draw AD; prolong BC to E; draw EA, and, parallel to it, DF.

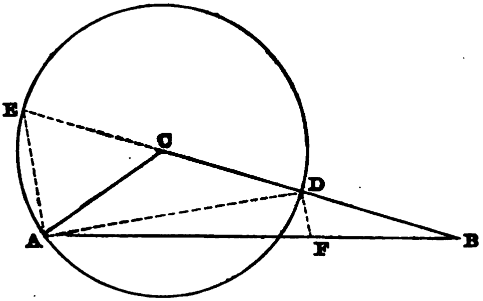

Fig. 61.

From the construction of the figure, it follows that the triangles ECA and ACD are isosceles; that EAD and ADF are right angles; and that the triangles AEB and FDB are similar.

$$\begin{aligned}\text{The angle } \mathrm{ADE} &= \tfrac{1}{2}\,\mathrm{ACE}\\ &= \tfrac{1}{2}\,(180^\circ - C)\\ &= \tfrac{1}{2}\,(A+B)\\ \text{The angle } \mathrm{DAF} &= A - \mathrm{CAD}\\ &= A - \mathrm{ADE}\\ &= A - \tfrac{1}{2}\,(A+B)\\ &= \tfrac{1}{2}\,(A-B)\end{aligned}$$

In the right triangles EAD and ADF,

$$\tan \tfrac{1}{2}\,(A+B) = \tan \mathrm{ADE} = \frac{\mathrm{AE}}{\mathrm{AD}}$$

and

$$\tan \tfrac{1}{2}\,(A-B) = \tan \mathrm{DAF} = \frac{\mathrm{FD}}{\mathrm{AD}}$$

whence, by division,

$$\frac{\tan \tfrac{1}{2}\,(A+B)}{\tan \tfrac{1}{2}\,(A-B)} = \frac{\mathrm{AE}}{\mathrm{AD}} \div \frac{\mathrm{FD}}{\mathrm{AD}} = \frac{\mathrm{AE}}{\mathrm{FD}} \quad \ldots\ldots \quad \text{(i)}$$

$$\begin{aligned}\text{The line } \mathrm{BE} &= \mathrm{BC} + \mathrm{CE} = \mathrm{BC} + \mathrm{CA} = a + b\\ \text{and } \mathrm{BD} &= \mathrm{BC} - \mathrm{CD} = \mathrm{BC} - \mathrm{CA} = a - b\end{aligned}$$

whence, by division,

$$\frac{a+b}{a-b}=\frac{\mathbf{BE}}{\mathbf{BD}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The similar triangles **AEB** and **FDB** give the proportion,

$$\frac{\mathbf{AE}}{\mathbf{FD}}=\frac{\mathbf{BE}}{\mathbf{BD}}$$

showing that the right-hand members of (i) and (ii) are equal. Therefore the left-hand members are equal, that is

$$\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

181. By adding the first two equations of [**117**] and dividing the result by the third equation, we get

$$\frac{a+b}{c}=\frac{\sin A+\sin B}{\sin C}.$$

The numerator of the second member of this equation is, by [**92**], equal to

$$2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).$$

The denominator, since $\sin C=\sin(180°-C)$ and $180°-C=A+B$, is equal to $\sin(A+B)$; and this last, by [**77**], is equal to

$$2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+B).$$

By substituting these quantities for the numerator and denominator, and cancelling equal factors, we get

$$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\mathbf{119}\,a]$$

Two other equations of the same form may be found by a cyclic change of letters

By subtracting the second equation of [117] from the first and dividing the result by the third, we get

$$\frac{a-b}{c}=\frac{\sin A-\sin B}{\sin C}$$

which is, by a process similar to the foregoing, reduced to the form,

$$\frac{a-b}{c}=\frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \quad . \quad . \quad . \quad . \quad [119\,b]^*$$

Two other equations of the same form.

The two formulas [119 *a*] and [119 *b*] can easily be proved geometrically from Figure 61. This is proposed as an exercise.

182. *The square of one side of a triangle is equal to the sum of the squares of the other two sides less twice the product of these sides multiplied by the cosine of the angle included by them.*

ANALYTIC PROOF. If the first equation of [**114**] be multiplied through by a, the second by $-b$, and the third by $-c$, and the three equations be then added, the result is

$$a^2-b^2-c^2=-2\,b\,c\cos A$$

If the first be multiplied through by $-a$, the second by b, and the third by $-c$, the result of adding is

$$-a^2+b^2-c^2=-2\,c\,a\cos B$$

If the first be multiplied through by $-a$, the second by $-b$, and the third by c, the result of adding is

$$-a^2-b^2+c^2=-2\,a\,b\cos C$$

---

* NOTE. If $b>a$, and therefore $B>A$, both numerators are negative. To make them positive the formula may be written

$$\frac{b-a}{c}=\frac{\sin \frac{1}{2}(B-A)}{\sin \frac{1}{2}(B+A)}$$

These results give the three equations,

$$\left.\begin{array}{l} a^2=b^2+c^2-2\,b\,c\cos A \\ b^2=c^2+a^2-2\,c\,a\cos B \\ c^2=a^2+b^2-2\,a\,b\cos C \end{array}\right\} \quad \ldots\ldots \quad [120]$$

which completely express the theorem.*

183. GEOMETRIC PROOF. Suppose the equation

$$a^2=b^2+c^2-2\,b\,c\cos A$$

is the one to be proved. There are two cases to be considered, one in which A is an acute angle, and one in which A is obtuse.

First, let A be acute. From one of the other vertices, as C, let fall a perpendicular upon the opposite side (Fig. 62), or upon the opposite side prolonged (Fig. 63). Then,

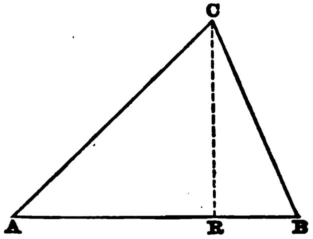

Fig. 62.

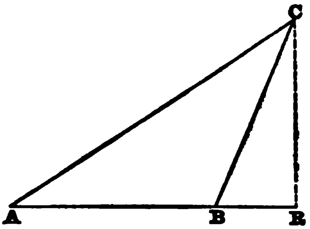

Fig. 63.

$$\text{(Fig. 62),} \qquad \mathrm{RB}=c-\mathrm{AR}$$
$$\text{(Fig. 63),} \qquad \mathrm{BR}=\mathrm{AR}-c$$

In either case, by squaring,

$$\overline{\mathrm{RB}}^2=c^2+\overline{\mathrm{AR}}^2-2c\times\mathrm{AR}.$$

---

* NOTE. The theorem of Pythagoras is a particular case of this theorem, as may be seen by making $A=90°$, or $B=90°$, or $C=90°$ in the formulas.

By adding $\overline{\mathrm{RC}}^2$ to each member of this, and remembering that

$$\overline{\mathrm{RB}}^2+\overline{\mathrm{RC}}^2=\overline{\mathrm{BC}}^2=a^2$$
$$\overline{\mathrm{AR}}^2+\overline{\mathrm{RC}}^2=\overline{\mathrm{AC}}^2=b^2$$

we get

$$a^2=c^2+b^2-2\,c\times \mathrm{AR}$$

But in both figures

$$\mathrm{AR}=b\cos A.$$

Therefore

$$a^2=c^2+b^2-2\,c\,b\cos A$$

which is the equation to be proved.

Secondly, let A be obtuse. The perpendicular from C (Fig. 64) falls without the triangle, and

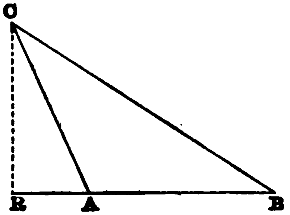

Fig. 64.

$$\mathrm{RB}=\mathrm{RA}+c$$

or, by squaring,

$$\overline{\mathrm{RB}}^2=\mathrm{RA}^2+c^2+2c\times \mathrm{RA}$$

Adding $\overline{\mathrm{RC}}^2$ to each member of this, and reducing as before, we get

$$a^2=b^2+c^2+2\,c\times \mathrm{RA}$$

But
$$\begin{aligned}\mathrm{RA}&=\mathrm{CA}\times\cos \mathrm{RAC}\\&=b\cos(180^\circ-A)\\&=-b\cos A\end{aligned}$$

Therefore, by substitution,

$$a^2=b^2+c^2-2\,c\,b\cos A$$

which, as before, is the equation to be proved.

The three equations just proved are by some writers regarded as the fundamental equations of plane triangles, and with good reason, for all others can be derived from them. It may be proposed as an exercise to derive from them the equations [114].*

184. The three equations [120], give the cosines of the three angles of a triangle in terms of the three sides. Thus,

$$\left.\begin{aligned}\cos A &= \frac{-a^2+b^2+c^2}{2\,b\,c}\\ \cos B &= \frac{a^2-b^2+c^2}{2\,c\,a}\\ \cos C &= \frac{a^2+b^2-c^2}{2\,a\,b}\end{aligned}\right\} \quad \ldots\ldots\ldots \quad [121]$$

When the values of the sides are small numbers, these equations give the angles quite easily by means of the natural cosines; but for large numbers requiring logarithmic work it is better to transform them so that the second member may be a product of factors and not a sum of terms.

To do this in the case of the first equation, subtract each member from 1, and reduce the second member as follows:

$$\begin{aligned}1-\cos A &= 1-\frac{-a^2+b^2+c^2}{2\,b\,c}\\ &= \frac{2\,b\,c+a^2-b^2-c^2}{2\,b\,c}\\ &= \frac{a^2-(b-c)^2}{2\,b\,c}\\ &= \frac{(a-b+c)\;(a+b-c)}{2\,b\,c}\end{aligned}$$

---

* If any two of the equations [120] be added, the result reduced is one of the equations [114].

But, by [84], the first member is equal to $2\sin^2\frac{1}{2}A$. Therefore,

$$2\sin^2\tfrac{1}{2}A=\frac{(a-b+c)\,(a+b-c)}{2\,b\,c}$$

or

$$\sin\tfrac{1}{2}A=\sqrt{\frac{(a-b+c)\,(a+b-c)}{4\,b\,c}}$$

To abbreviate this, let $2\,s$ denote the sum of the sides of the triangle, so that

$$\left.\begin{aligned}2s&=a+b+c\\ 2\,(s-a)&=-a+b+c\\ 2\,(s-b)&=a-b+c\\ 2\,(s-c)&=a+b-c\end{aligned}\right\}\quad\dots\dots\quad[122]$$

Then, by substitution,

$$\sin\tfrac{1}{2}A=\sqrt{\frac{2\,(s-b)\times 2\,(s-c)}{4\,b\,c}}$$

which, reduced, is the first equation of the following set, the others being derived from [121] in the same way:

$$\left.\begin{aligned}\sin\tfrac{1}{2}A&=\sqrt{\frac{(s-b)\,(s-c)}{b\,c}}\\ \sin\tfrac{1}{2}B&=\sqrt{\frac{(s-c)\,(s-a)}{c\,a}}\\ \sin\tfrac{1}{2}C&=\sqrt{\frac{(s-a)\,(s-b)}{a\,b}}\end{aligned}\right\}\quad\dots\dots\quad[123]$$

185. Another transformation of the first equation of [121] is made by adding each member to 1, and reducing the second member, as follows:

$$1+\cos A=1+\frac{-a^2+b^2+c^2}{2\,b\,c}$$

$$=\frac{2\,b\,c+b^2+c^2-a^2}{2\,b\,c}$$

$$=\frac{(b+c)^2-a^2}{2\,b\,c}$$

$$=\frac{(a+b+c)\ (-a+b+c)}{2\,b\,c}$$

But, by [83], $1+\cos A=2\cos^2\frac{1}{2}A$.

Therefore

$$2\cos^2\tfrac{1}{2}A=\frac{(a+b+c)\ (-a+b+c)}{2\,b\,c}$$

or

$$\cos\tfrac{1}{2}A=\sqrt{\frac{(a+b+c)\ (-a+b+c)}{4\,b\,c}}$$

which, by the abbreviations [122], becomes the first equation of the following set:

$$\left.\begin{aligned}\cos\tfrac{1}{2}A&=\sqrt{\frac{s\,(s-a)}{b\,c}}\\ \cos\tfrac{1}{2}B&=\sqrt{\frac{s\,(s-b)}{c\,a}}\\ \cos\tfrac{1}{2}C&=\sqrt{\frac{s\,(s-c)}{a\,b}}\end{aligned}\right\}\quad .\ .\ .\ .\ .\quad [124]$$

186. Dividing each equation of [123] by the corresponding one of [124], we have

$$\left.\begin{aligned}\tan\tfrac{1}{2}A&=\sqrt{\frac{(s-b)\ (s-c)}{s\,(s-a)}}\\ \tan\tfrac{1}{2}B&=\sqrt{\frac{(s-c)\ (s-a)}{s\,(s-b)}}\\ \tan\tfrac{1}{2}C&=\sqrt{\frac{(s-a)\ (s-b)}{s\,(s-c)}}\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\quad [125]$$

187. If we put, for abbreviation,

$$r=\sqrt{\frac{(s-a)\ (s-b)\ (s-c)}{s}} \quad \ldots\ldots\ [126]$$

the last set of equations becomes

$$\left.\begin{aligned}\tan \tfrac{1}{2} A&=\frac{r}{s-a}\\ \tan \tfrac{1}{2} B&=\frac{r}{s-b}\\ \tan \tfrac{1}{2} C&=\frac{r}{s-c}\end{aligned}\right\} \quad \ldots\ldots\ [127]$$

188. The geometrical meaning of some of the foregoing expressions may be learned by inscribing in the triangle A B C (Fig. 65) a circle, touching the sides at points D, E, and F.

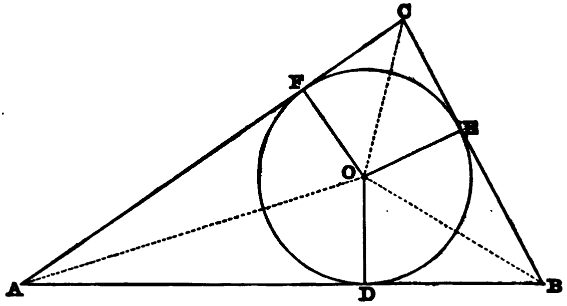

Fig. 65.

Since the two tangents drawn from a point to a circle are equal, we have,

A D = A F, B E = B D, C F = C E,

and, by adding,

A D + B E + C F = A F + B D + C E

Each member of this equation is equal to one-half the perimeter of the triangle, because both members together make the whole perimeter. But one-half of the perimeter is $\frac{1}{2}(a+b+c)$, or s. Therefore,

$$\left.\begin{aligned} s &= \mathrm{AD}+\mathrm{BE}+\mathrm{CF} \\ s &= \mathrm{AF}+\mathrm{BD}+\mathrm{CE} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad [\mathrm{i}]$$

From the figure, again, we have

$$\begin{aligned} a &= \mathrm{BE}+\mathrm{CE} \\ b &= \mathrm{CF}+\mathrm{AF} \\ c &= \mathrm{AD}+\mathrm{BD} \end{aligned}$$

or, substituting **C F** for its equal **C E**,
A D for its equal **A F**,
B E for its equal **B D**,

$$\left.\begin{aligned} a &= \mathrm{BE}+\mathrm{CF} \\ b &= \mathrm{CF}+\mathrm{AD} \\ c &= \mathrm{AD}+\mathrm{BE} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (\mathrm{ii})$$

By subtracting each of the equations (ii) from the first equation of (i) we get

$$\begin{aligned} s-a &= \mathrm{AD} \\ s-b &= \mathrm{BE} \\ s-c &= \mathrm{CF} \end{aligned}$$

In a similar way it may be shown that

$$\begin{aligned} s-a &= \mathrm{AF} \\ s-b &= \mathrm{BD} \\ s-c &= \mathrm{CE} \end{aligned}$$

189. GEOMETRIC PROOF OF [127]. By drawing the radii **O D**, **O E**, and **O F** (Fig. 65), also the lines **O A**, **O B**, and **O C**, which bisect the angles *A*, *B*, and *C* respectively, we have, in the right triangles **A D O**, **B E O**, and **C F O**,

$$\tan\tfrac{1}{2}A=\frac{\mathrm{OD}}{\mathrm{AD}},\quad \tan\tfrac{1}{2}B=\frac{\mathrm{OE}}{\mathrm{BE}},\quad \tan\tfrac{1}{2}C=\frac{\mathrm{OF}}{\mathrm{CF}},$$

or, substituting for each numerator r, to denote the radius of the inscribed circle, and for each denominator the value found in the last article,

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}, \quad \tan \tfrac{1}{2} B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

We thus discover that r, which [126] was adopted as an abbreviation for the quantity

$$\sqrt{\frac{(s-a)\ (s-b)\ (s-c)}{s}}$$

is numerically equal to the radius of the inscribed circle.

AREA OF TRIANGLES.

190. If perpendiculars AP, BQ, CR be drawn from the vertices of a triangle to the opposite sides respectively, the area of the triangle will be expressed by either of the following products:

$$\tfrac{1}{2} a \times \mathrm{AP}, \quad \tfrac{1}{2} b \times \mathrm{BQ}, \quad \tfrac{1}{2} c \times \mathrm{CR}.$$

But, by projecting the sides upon the perpendicular, we find

$$\begin{aligned} \mathrm{AP} &= b \sin C = c \sin B \\ \mathrm{BQ} &= c \sin A = a \sin C \\ \mathrm{CR} &= a \sin B = b \sin A \end{aligned}$$

Therefore, by substitution,

$$\left.\begin{array}{l}\text{Area of}\\ \text{triangle}\end{array}\right\} = \tfrac{1}{2}\, a\, b \sin C = \tfrac{1}{2}\, b\, c \sin A = \tfrac{1}{2}\, c\, a \sin B. \qquad [128]$$

which gives the area in terms of two sides and the sine of the included angle.

191. If, in the formula just proved, we substitute for b in the first product, for c in the second, and for a in the third, their respective equivalents,

$$\frac{a \sin B}{\sin A}, \quad \frac{b \sin C}{\sin B}, \text{ and } \frac{c \sin A}{\sin C}$$

taken from [115], the result

$$\left.\begin{matrix}\text{Area of}\\ \text{triangle}\end{matrix}\right\} = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{b^2 \sin C \sin A}{2 \sin B} = \frac{c^2 \sin A \sin B}{2 \sin C} \quad [129]$$

gives the area in terms of one side, and the sines of the angles.

192. If, in this last formula, we substitute for a^2, b^2, and c^2 their equivalents,

$$4\,R^2 \sin^2 A, \qquad 4\,R^2 \sin^2 B, \qquad 4\,R^2 \sin^2 C,$$

taken from [117], all three products reduce to the same,

$$\left.\begin{matrix}\text{Area of}\\ \text{triangle}\end{matrix}\right\} = 2\,R^2 \sin A \sin B \sin C \;.\;.\;.\;.\quad [130]$$

giving the area in terms of the radius of the circumscribed circle and the sines of the angles.

193. If, in this last formula, we substitute for the sines of the angles their equivalents,

$$\frac{a}{2\,R}, \quad \frac{b}{2\,R}, \text{ and } \frac{c}{2\,R},$$

taken from [117] the result,

$$\left.\begin{matrix}\text{Area of}\\ \text{triangle}\end{matrix}\right\} = \frac{a\,b\,c}{4\,R} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\quad [131]$$

shows that the area of a triangle is equal to the product of its three sides divided by four times the radius of the circumscribed circle.

194. It is proved in geometry, that the area of a plane triangle is equal to one-half the perimeter multiplied by the radius of the inscribed circle; that is, in the symbols above used, [122] and [126].

$$\left.\begin{array}{l}\text{Area of}\\ \text{triangle}\end{array}\right\}=s\,r \quad \cdots\cdots\cdots \quad [132]$$

or, substituting for r its equivalent [126],

$$\left.\begin{array}{l}\text{Area of}\\ \text{triangle}\end{array}\right\}=\sqrt{s\,(s-a)\,(s-b)\,(s-c)} \quad \cdots \quad [133]$$

which gives the area in terms of the three sides.

195. By combining [131] with [133] we get

$$2\,R=\frac{a\,b\,c}{2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}\,. \quad \cdots\cdots \quad [134]$$

which gives the modulus of the triangle in terms of the three sides.

SOLUTIONS OF TRIANGLES.

I.

196. *Given one side and two angles of a triangle, to find the other parts.*

The third angle is found at once from the equation

$$A+B+C=180°$$

The rest of the solution is by means of [117]

$$a=2\,R\sin A$$
$$b=2\,R\sin B$$
$$c=2\,R\sin C$$

The logarithm of $2\,R$ is first found by means of the equation containing the given side and the opposite angle. Then the other two equations give the other two sides.

197. Example. Given the side $b = 124.6$, the angle $A = 59° \ 17' \ 10''$, and the angle $C = 74° \ 34' \ 20''$.

$$\begin{array}{r}
A = 59° \ 17' \ 10'' \\
\underline{C = 74° \ 34' \ 20''} \\
133° \ 51' \ 30'' \\
B = \ \ 46° \ \ 8' \ 30''
\end{array}$$

$$\begin{array}{lrlr}
\log b & = 2.09552 & & \\
\log \sin B & = \underline{9.85797} & & \\
\log 2R & = 2.23755 & \log 2R & = 2.23755 \\
\log \sin A & = \underline{9.93436} & \log \sin C & = \underline{9.98406} \\
\log a & = 2.17191 & \log c & = 2.22161 \\
a & = 148.56 & c & = 166.58
\end{array}$$

198. Another solution is by means of the formulas [119 *a*] and [119 *b*], which, by a cyclic change of letters, become, for the above example,

$$\frac{c+a}{b} = \frac{\cos \frac{1}{2}(C-A)}{\cos \frac{1}{2}(C+A)}, \qquad \frac{c-a}{b} = \frac{\sin \frac{1}{2}(C-A)}{\sin \frac{1}{2}(C+A)}.$$

$$\begin{array}{ll}
C-A = 15° \ 17' \ 10'', & \frac{1}{2}(C-A) = 7° \ 38' \ 35'', \\
C+A = 133° \ 51' \ 30'', & \frac{1}{2}(C+A) = 66° \ 55' \ 45'',
\end{array}$$

$$\begin{array}{rr rr}
\log \cos \frac{1}{2}(C-A) = & 9.99612 & \log \sin \frac{1}{2}(C-A) = & 9.12386 \\
\log \cos \frac{1}{2}(C+A) = & \underline{9.59314} & \log \sin \frac{1}{2}(C+A) = & \underline{9.96380} \\
& 0.40298 & & 9.16006 \\
\log b = & \underline{2.09552} & \log b = & \underline{2.09552} \\
\log (c+a) = & 2.49850 & \log (c-a) = & 1.25558 \\
c+a = & 315.136 & c-a = & 18.013 \\
c-a = & \underline{18.013} & c+a = & \underline{315.136} \\
& 297.123 & & 333.149 \\
a = & 148.56 & c = & 166.57
\end{array}$$

The two solutions, being independent, may be used as checks on each other.

II.

199. *Given two sides and the included angle of a triangle, to find the other parts.*

Let a, b, and C be the given parts.

The value of $\frac{1}{2}(A+B)$ is one-half the remainder found by subtracting C from 180°.

The value of $\frac{1}{2}(A-B)$ is computed by means of the equation

$$\tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A+B)$$

taken from [**118**].

The sum of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ is the angle A, and the difference is the angle B.

The third side, may be found by means of either of the equations,

$$c = \frac{a \sin C}{\sin A} = \frac{b \sin C}{\sin B}$$

taken from [**115**]; or, by means of either of the equations,

$$c = (a+b)\frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = (a-b)\frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)}$$

taken from [**119***a*] and [**119***b*].

200. EXAMPLE. The two sides of a triangle are 347.2 and 485.7, and the angle included by them is 53° 47′ 40″. What are the other parts?

In choosing the symbols, we let a represent the greater given side, so that the angle A may be greater than the angle B, making $\frac{1}{2}(A-B)$ as well as $a-b$ positive.

$$\begin{array}{rl}
a = & 485.7 \\
b = & 347.2 \\
a-b = & 138.5 \\
a+b = & 832.9 \\
C = & 53^\circ\ 47'\ 40'' \\
A+B = & 126^\circ\ 12'\ 20'' \\
\frac{1}{2}(A+B) = & 63^\circ\ \ 6'\ 10'' \\
\frac{1}{2}(A-B) = & 18^\circ\ \ 8'\ 59'' \\
A = & 81^\circ\ 15'\ \ 9'' \\
B = & 44^\circ\ 57'\ 11'' \\
 & \\
c = & 396.53
\end{array}$$

$$\begin{array}{ll}
\log (a-b) & = 2.14145 \\
\text{co} \log (a+b) & = 7.07941 \\
\log \tan \frac{1}{2}(A+B) & = \underline{10.29476} \\
\log \tan \frac{1}{2}(A-B) & = \underline{9.51562} \\
\log a & = 2.68637 \\
\log \sin C & = 9.90682 \\
\text{co} \log \sin A & = \underline{0.00508} \\
\log c & = \underline{2.59827} \\
\log b & = 2.54058 \\
\log \sin C & = 9.90682 \\
\text{co} \log \sin B & = \underline{0.15087} \\
\log c & = 2.59827
\end{array}$$

The agreement between the two values of c here found is a proof of the work. A further and independent proof is found in computing c by [119 *a*] or [119 *b*].

$$\begin{array}{ll}
\log (a+b) & = 2.92059 \\
\log \cos \frac{1}{2}(A+B) & = 9.65552 \\
\text{co} \log \cos \frac{1}{2}(A-B) & = \underline{0.02216} \\
\log c & = 2.59827
\end{array}$$

$$\begin{array}{ll}
\log (a-b) & = 2.14145 \\
\log \sin \frac{1}{2}(A+B) & = 9.95028 \\
\text{co} \log \sin \frac{1}{2}(A-B) & = \underline{0.50654} \\
\log c & = 2.59827
\end{array}$$

III.

201. *Given two sides of a triangle and the angle opposite one of them, to find the other parts.*

Let a, b, and A be the given parts.

The angle opposite the other given side is found by means of the equations,

$$2R = \frac{a}{\sin A} \qquad \sin B = \frac{b}{2R}$$

taken from [117]. Two supplementary values of B are found, corresponding to the two geometrical solutions which can be made of this problem. See Art. 70. Let B_1 be the acute, and B_2 the obtuse value of the angle B. The third angle has two values, one found by subtracting $A + B_1$, and the other by subtracting $A + B_2$, from 180°, thus:

$$C_1 = 180° - (A + B_1), \qquad C_2 = 180° - (A + B_2)$$

Since $B_1 = 180° - B_2$, and $B_2 = 180° - B_1$, these equations become, by substitution,

$$C_1 = B_2 - A \qquad C_2 = B_1 - A$$

The third side will have two values, which are found by means of the equations,

$$c_1 = 2\,R \sin C_1, \qquad c_2 = 2\,R \sin C_2$$

taken from [117].

If sin B comes out greater than 1, there is no solution. If C_2 comes out negative, the second solution is inadmissible. All the special cases described in Art. 76 will be indicated by the nature of the results obtained from the above formulas.

202. Examples.

1. Given two sides of a triangle, 36.74 and 39.82, and the angle opposite the former, 49° 36′ 30″, to find the other parts.

$$\begin{array}{lllll}
a = 36.74 & & \log a & = & 1.56514 \\
b = 39.82 & & \log \sin A & = & \underline{9.88174} \\
A = 49° \; 36' \; 30'' & & \log 2\,R & = & \underline{1.68340} \\
 & & \text{co} \log 2\,R & = & 8.31660 \\
 & & \log b & = & \underline{1.60010} \\
 & & \log \sin B & = & 9.91670
\end{array}$$

$$\begin{array}{rlrl}
B_1 = & 55° \; 38' \; 8'' & B_2 = & 124° \; 21' \; 52'' \\
A = & \underline{49° \; 36' \; 30''} & A = & \underline{49° \; 36' \; 30''} \\
C_2 = & 6° \; 1' \; 38'' & C_1 = & 74° \; 45' \; 22''
\end{array}$$

$$\begin{array}{llll}
\log \sin C_2 = 9.02119 & & \log \sin C_1 = & 9.98444 \\
\log \; 2\,R = \underline{1.68340} & & \log 2\,R = & \underline{1.68340} \\
\log c_2 = 0.70459 & & \log c_1 = & 1.66784 \\
c_2 = 5.0651 & & c_1 = & 46.541
\end{array}$$

2. Given one side of a triangle $=39$, another $=57$, and the angle opposite the former $=47^\circ\ 23'$, to find the other parts.

$$\begin{array}{ll}
a = 39 & \log a = 1.59106 \\
b = 57 & \log \sin A = \underline{9.86682} \\
A = 47^\circ\ 23' & \log 2R = \underline{1.72424} \\
 & \text{co}\log 2R = 8.27576 \\
 & \log b = \underline{1.75587} \\
 & \log \sin B = 0.03163
\end{array}$$

No solution: $\sin B > 1$.

3. Given one side of a triangle $=322$, another $=283$, and the angle opposite the former $=57^\circ\ 25'\ 15''$, to find the other parts.

$$\begin{array}{ll}
a = 322 & \log a = 2.50786 \\
b = 283 & \log \sin A = \underline{9.92565} \\
A = 57^\circ\ 25'\ 15'' & \log 2R = \underline{2.58221} \\
 & \text{co}\log 2R = 7.41779 \\
 & \log b = \underline{2.45179} \\
 & \log \sin B = 9.86958 \\
B_1 = 47^\circ\ 46'\ 55'' & B_2 = 132^\circ\ 13'\ 5'' \\
A = \underline{57^\circ\ 25'\ 15''} & A = \underline{57^\circ\ 25'\ 15''} \\
C_2,\ \text{negative.} & C_1 = 74^\circ\ 47'\ 50'' \\
 & \log \sin C_1 = 9.98452 \\
 & \log 2R = \underline{2.58221} \\
 & \log c_1 = 2.56673 \\
 & c_1 = 368.75
\end{array}$$

4. Given one side of a triangle $=48.2$, another $=25.8$, and the angle opposite the former $=137^\circ\ 43'$, to find the other parts.

Results:

$$\begin{array}{ll}
B_1 = 21^\circ\ 6'\ 27'' & B_2 = 158^\circ\ 53'\ 33'' \\
C_2,\ \text{negative.} & C_1 = 21^\circ\ 10'\ 33'' \\
 & c_1 = 25.880
\end{array}$$

IV.

203. *Given the three sides of a triangle to find the angles.*

The solution is by means of the formulas [**123**] or [**124**] or [**127**], of which the last are generally the most convenient.

204. Example. Given the sides of a triangle 56.3, 62.9, and 59.4, to find the angles.

$a=56.3$	co log s	$=8.04915$
$b=62.9$	log $(s-a)$	$=1.51851$
$c=59.4$	log $(s-b)$	$=1.42160$
178.6	log $(s-c)$	$=1.47567$
$s=89.3$	log r^2	$=2.46493$
$s-a=33.0$	log r	$=1.23246$
$s-b=26.4$	log $(s-a)$	$=1.51851$
$s-c=29.9$	log tan $\frac{1}{2}A$	$=9.71395$
$A=54° 43' 38''$	$\frac{1}{2}A=27° 21' 49''$*	
	log r	$=1.23246$
	log $(s-b)$	$=1.42160$
	log tan $\frac{1}{2}B$	$=9.81086$
$B=65° 48' 2''$	$\frac{1}{2}B=32° 54' 1''$	
	log r	$=1.23246$
	log $(s-c)$	$=1.47567$
	log tan $\frac{1}{2}C$	$=9.75679$
$C=59° 28' 14''$	$\frac{1}{2}C$	$=29° 44' 7''$
$179° 59' 54''$		

---

[*] Since the last figure of log r might with equal accuracy have been 7, the last figure of log tan $\frac{1}{2}A$ might have been 6, giving 50″ in the angle. The mean between 48″ and 50″ is here taken. The same is done in finding $\frac{1}{2}B$ and $\frac{1}{2}C$.

The sum of the three angles computed should be 180°, but may vary slightly from that in consequence of the omission of figures beyond the fifth decimal place in the logarithms. In this example, the possible error in each of the first four logarithms is one half of a unit in the last place of decimals; then the possible error in log r is ± 1, and the possible error in each of the log tangents is ± $1\frac{1}{2}$ in the fifth decimal place. Therefore the possible error in each of the half angles, found by consulting the tables, is about $\pm 3''$; in each of the whole angles, $\pm 6''$; and in the sum of the three, $\pm 18''$. The actual error in the present example is $+6''$.

205. Exercises.

For exercises in solving triangles by the methods above given, turn to Art. 88. Taking as data three parts of any triangle there found, compute the other three parts. Since each triangle furnishes ten sets of given parts, the forty furnish in all four hundred examples.

The following exercises suggest simple arithmetical illustrations of the formulas, and are readily worked by the three-place values of natural functions in Table I.

1. The sides of a triangle measuring 5, 6, and 8 respectively, substitute these values for a, b, and c in [114], solve the equations algebraically for the values of the cosines of the three angles, and then find the angles by Table I.

2. What are the results when the sides are so given that one of them is greater than the sum of the other two, as, for example, when $a=9$, $b=4$, and $c=3$?

3. What are the results when the sides are so given that one of them is equal to the sum of the other two, as, for example, when $a=8$, $b=5$, and $c=3$?

4. The sides of a triangle measuring 12, 15, and 18, compute the modulus by [134], the sines of the angles by [117], and then find the angles from Table I.

5. The sides of a triangle measuring 9, 10, and 11, find the angles from their cosines [121].

6. The sides of a triangle measuring 5, 6, and 7, compute the angles by [123].

7. The sides of a triangle measuring 3, 4, and 6, compute the angles by [127].

8. Find the radius of the circle inscribed in, and of the circle circumscribed about a triangle whose sides are 10, 15, and 20; by [126], [131], and [132].

9. One side of a triangle measures 16 inches, and the adjacent angles 52° and 74°. Find the other parts. [117.]

10. Two sides of a triangle measure 10 and 14 inches, and the included angle 84°. Find the other parts by [118], [119 *a*], and [119 *b*].

11. Two sides of a triangle measure 15 and 20 inches, and the angle opposite the former 37°. Find the other parts. [117.]

12. The diameter of a circle is 2 feet, and the angles of a triangle inscribed in it are 49°, 63°, and 68°. Find the area of the triangle by [130], the sides by [117], and the area again by [133].

FORMULAS FOR SPECIAL CASES.

206. An angle is not determined with equal accuracy in all parts of the quadrant by means of the same function, nor in the same part of the quadrant by means of different functions. For example, a small acute angle is determined much more accurately by means of its sine than by means of its cosine, because in that part of the quadrant the variations of the former are much greater than are the variations of the latter for the same variations in the angle.

For a similar reason, an angle near to 90° is more accurately determined by means of its cosine than by means of its sine.

It is sometimes desirable, therefore, to change a formula by introducing a different function or a different angle. The following are the more important instances that may occur.

207. When the hypothenuse h and one leg a of a right triangle are nearly equal, the angle A is near 90°, and cannot be accurately determined from its sine; also the angle B is near 0°, and cannot be accurately determined from its cosine. The formula

$$\sin A = \cos B = \frac{a}{h}$$

which was used in Arts. 33 and 48, should therefore be changed. This is done by substituting for $\cos B$ its equivalent $1-2\sin^2\frac{1}{2}B$ taken from [**84**]. Then

$$1-2\sin^2\tfrac{1}{2}B = \frac{a}{h}$$

whence

$$\sin\tfrac{1}{2}B = \sqrt{\frac{h-a}{2h}} \quad \text{. . . . . . .} \quad [\mathbf{135}]$$

which gives small values of B accurately.

From this formula, by subtracting the square of each member from 1, we get

$$\cos^2\tfrac{1}{2}B = 1 - \frac{h-a}{2h} = \frac{h+a}{2h},$$

or

$$\cos\tfrac{1}{2}B = \sqrt{\frac{h+a}{2h}}$$

Dividing [**135**] by this, we get

$$\tan \tfrac{1}{2} B = \sqrt{\frac{h-a}{h+a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [136]$$

or, multiplying numerator and denominator of the fraction under the radical by $h-a$,

$$\tan \tfrac{1}{2} B = \frac{h-a}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [137]$$

two other formulas giving accurate values of B when h and a are nearly equal.

208. When the two legs, and consequently the two oblique angles of a right triangle are nearly equal, their differences can be accurately computed by a formula derived as follows: [See Appendix C.]

Since

$$90° - B = A,$$

we have, by adding A to each member,

$$90° - (B - A) = 2\,A$$

and

$$\sin (B - A) = \cos [90° - (B - A)] = \cos 2\,A.$$

But by [70],

$$\cos 2\,A = \cos^2 A - \sin^2 A$$

or, by substituting in the second member,

$$\frac{b}{h} = \cos A \qquad \text{and } \frac{a}{h} = \sin A,$$

$$\cos 2\,A = \frac{b^2}{h^2} - \frac{a^2}{h^2} = \frac{(b+a)\,(b-a)}{h^2}$$

Therefore

$$\sin (B - A) = \frac{(b+a)\,(b-a)}{h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [138]$$

Again,

$$\cos (B - A) = \sin [90° - (B - A)] = \sin 2\,A$$

But by [69]

$$\sin 2A = 2 \sin A \cos A$$

or, by substituting as above,

$$\sin 2A = \frac{2ab}{h^2}$$

Therefore

$$\cos (B-A) = \frac{2ab}{h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [139]$$

The quotient of [138] divided by [139] is

$$\tan (B-A) = \frac{(b+a)(b-a)}{2ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad [140]$$

a formula for right triangles by which $B-A$ can be very accurately computed when a and b are given nearly equal, or $b-a$ very accurately when A and B are given nearly equal.

209. In an oblique triangle, when two angles, as A and B, are nearly equal, the opposite sides are also nearly equal, but the differences $a-b$ and $A-B$ can be computed the one from the other very accurately by formulas derived as follows:

By [115],

$$b = \frac{a \sin B}{\sin A};$$

therefore

$$a-b = a - \frac{a \sin B}{\sin A} = \frac{a (\sin A - \sin B)}{\sin A}$$

which by [93] reduces to the form

$$a-b = \frac{2a \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B)}{\sin A} \quad . \quad . \quad . \quad [141]$$

or

$$\sin \tfrac{1}{2}(A-B) = \frac{(a-b) \sin A}{2a \cos \tfrac{1}{2}(A+B)} \quad . \quad . \quad . \quad . \quad . \quad . \quad [142]$$

210. ACCURATE COMPUTATION OF SMALL ANGLES.

In a circle whose radius is 1 the arcs subtending small angles at the centre are very nearly equal in numerical value to the sines and to the tangents of those angles. The smaller the angle the more nearly equal are the arc, the sine, and the tangent. In other words, the ratio

$$\frac{\text{sine of } A}{\text{arc of } A}$$

has a value but slightly less than 1, and the ratio

$$\frac{\text{tangent of } A}{\text{arc of } A}$$

has a value but slightly greater than 1, when A is a small angle.

211. These facts may be learned from the tables of natural functions,* or by computing a few values of the natural functions from their logarithms.

Thus, with seven-place tables we find,

ANGLE.	ARC.	SINE.	TANGENT.
1′	0.00029 08882	0.00029 08882	0.00029 08882
10′	0.00290 8882	0.00290 8878	0.00290 8890
60′	0.01745 3293	0.01745 2406	0.01745 5065
100′	0.02908 882	0.02908 472	0.02909 703
120′	0.03490 659	0.03489 950	0.03492 077
180′	0.05235 988	0.05233 596	0.05240 778

If decimals beyond the fifth place are rejected, the differences between the arc, the sine, and the tangent will not begin to appear until the angle reaches 60′, and will not exceed a unit in the fifth decimal place before the angle reaches 120′ or 2°. Even at 180′ or 3° the differences remain quite small.

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* Log. Trig. Tab., pp. 48 and 142.

212. In a circle whose radius is 1, the length of the arc measuring 180° is π. The length of the arc measuring $1''$ is therefore $\frac{\pi}{180 \times 60 \times 60}$, or $\frac{\pi}{648000}$.

If the arc measuring any angle A be divided by the arc measuring $1''$, the quotient is the number of seconds in the angle A. Denoting this number of seconds by A'', we have

$$A'' = \frac{\text{arc } A}{\text{arc } 1''} = \frac{\text{arc } A \times 648000}{\pi} \quad . \quad . \quad . \quad . \quad [143]$$

213. The ratios

$$\frac{\sin A}{\text{arc } A} \quad \text{and} \quad \frac{\tan A}{\text{arc } A}$$

are commonly used, not in these forms, but in forms obtained by multiplying each of them by $\frac{\pi}{648000}$, or arc $1''$. This gives

$$\frac{\sin A \times \text{arc } 1''}{\text{arc } A} \quad \text{and} \quad \frac{\tan A \times \text{arc } 1''}{\text{arc } A},$$

which, reduced by [143] become

$$\frac{\sin A}{A''} \quad \text{and} \quad \frac{\tan A}{A''}.$$

214. Tables giving the values of

$$\log\left(\frac{\sin A}{A''}\right) \quad \text{and} \quad \log\left(\frac{\tan A}{A''}\right)$$

at convenient intervals have been prepared, and are to be found in most of the larger books of logarithmic tables. For a five-place table of this kind, see Log. Trig. Tab., p. 140.

215. For shortness put

$$\mathrm{S} = \log\left(\frac{\sin A}{A''}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad [144]$$

and

$$\mathrm{T} = \log\left(\frac{\tan A}{A''}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad [145]$$

Then

$$\mathrm{S} = \log \sin A - \log A''$$
$$\mathrm{T} = \log \tan A - \log A''$$

whence

$$\log A'' = \log \sin A - \mathrm{S} \quad . \; . \; . \; . \; . \; . \; . \quad [146]$$
$$\log A'' = \log \tan A - \mathrm{T} \quad . \; . \; . \; . \; . \; . \; . \quad [147]$$

two formulas for finding the angle very accurately when the logarithms of the functions are given; and

$$\log \sin A = \log A'' + \mathrm{S} \quad . \; . \; . \; . \; . \; . \; . \; . \quad [148]$$
$$\log \tan A = \log A'' + \mathrm{T} \quad . \; . \; . \; . \; . \; . \; . \; . \quad [149]$$

two formulas for finding the functions very accurately when the angle is given.

216. When the ratios $\frac{\sin A}{\text{arc } A}$ and $\frac{\tan A}{\text{arc } A}$, because of the extreme smallness of the angle A, are extremely near 1 in value, then the ratios

$$\frac{\sin A}{A''} \quad \text{and} \quad \frac{\tan A}{A''}$$

are extremely near $\frac{\pi}{648000}$, or arc $1''$, in value. The logarithm of this value ($= \log \pi - \log 648000$) is $\bar{6}.68557$ or, adding 10 to the characteristic, 4.68557. Therefore, for the smallest angles we have $\mathrm{S} = 4.68557$ and $\mathrm{T} = 4.68557$.

So slowly do the ratios change that the last figure of **S** does not change from 7 to 6 until the angle reaches 2409″, nor from 6 to 5 until it reaches 3417″, or nearly 1°; and the whole change as the angle passes from 0° to 2° is from 4.68557 to 4.68549, a diminution of 0.00008. The change in the value of **T** is a little more rapid, the whole change as the angle passes from 0° to 2° being from 4.68557 to 4.68575, an increase of 0.00018. See Log. Trig. Tab., p. 140.

217. When tables of the values of **S** and **T** are not within reach, the following equations may be used to compute very closely any needed values:

$$\mathbf{S}=4.6855749-\tfrac{1}{3}(10-\log\cos A) \quad . \quad . \quad . \quad . \quad [150]$$

$$\mathbf{T}=4.6855749+\tfrac{2}{3}(10-\log\cos A) \quad . \quad . \quad . \quad . \quad [151]$$

218. Examples.

I. Given in a right triangle $h=5612.85$ and $a=5612.09$, to find the included angle.

By [135] and [146] with the special table for **S**,

$$\begin{array}{lrr}
\log(h-a) & = & 9.88081 \\
\text{colog } 2h & & \underline{5.94979} \\
 & 2) & \underline{15.83060} \\
\log\sin\tfrac{1}{2}B & = & 7.91530 \\
\mathbf{S} & = & \underline{4.68557} \\
\log\tfrac{1}{2}B'' & = & 3.22973 \\
\tfrac{1}{2}B'' & = & 1697.2'' \\
B & = & 56'\ 34.4''
\end{array}$$

If B had been computed from its cosine, or, what is the same thing, if A had been computed from its sine, as in Art. 49, the result would have been

$$\log\cos B = 9.99994$$

and

B anywhere from 55′ to 59′.

II. Given $\log\tan A = 6.57840$ to find A.

$$\begin{array}{lrr}
\log\tan A & = & 6.57840 \\
\mathbf{T} & = & \underline{4.68557} \\
\log A'' & & 1.89283
\end{array}$$

$$A=78.13''\text{ or }1'\ 18.13''$$

By ordinary interpolation the (less accurate) result would have been 1′ 22.8″.

CHAPTER VIII.

GENERAL PROPERTIES OF SPHERICAL TRIANGLES.

219. A spherical triangle is the figure formed on the surface of a sphere by three arcs of great circles joining three points.

Let **A**, **B**, and **C** be three points on the surface of a sphere. A plane passed through two of these points, as **A** and **B**, and the centre of the sphere cuts the surface in the circumference of a great circle. This circumference is divided by the points **A** and **B** into two arcs which, in general, are unequal, one being less and one greater than a semi-circumference. Likewise, the points **B** and **C** divide the circumference of the great circle passing through them into two arcs, one less and one greater than a semi-circumference; and the same is true of the two points **C** and **A**. Thus there are six arcs of great circles joining the three points **A**, **B**, and **C**. Three of these arcs are less and three are greater than semi-circumferences. Any three of these six arcs forming a closed path from one point to another, from that to the third, and thence back to the first form a spherical triangle according to the definition.

Among the different forms of spherical triangle thus included under the definition, there is one which has each of its sides less than a semi-circumference. This is the only form considered in geometry; and to this, for the present, our attention shall be limited. Later we shall see how the solutions of all other forms are related to the solution of this, the fundamental form.

220. The sides of a spherical triangle are the measures of the angles **A O B**, **B O C**, and **C O A**, formed by the radii drawn from the centre of the sphere to the vertices of the triangle. Accordingly, the sides are usually expressed in angular measure, that is, in degrees, minutes, and seconds.

221. An angle of a spherical triangle is the angle formed by a pair of tangent straight lines drawn at a vertex of the triangle, one tangent in the plane of one great circle and the other in the plane of the other; for, at the point of tangency, the sides of the triangle have the same directions as these tangents.

222. The angles of a spherical triangle are the measures of the wedge-angles formed by the planes of the sides; for a wedge-angle is measured by the angle between two lines perpendicular to its edge at the same point, one in each face; and the two tangents forming the angle of the spherical triangle are two such perpendiculars.

223. Thus the relations between the sides and angles of a spherical triangle are the relations between the line-angles and the wedge-angles of a three-faced solid angle having its vertex at the centre of the sphere.

224. The sides of a spherical triangle will be denoted by a, b, and c, and the angles by A, B, and C.

THE FUNDAMENTAL FORMULAS.

225. Let **A B C** be any spherical triangle whose sides and angles are each less than 180°, Figs. 66, 67, and 68. At **A** draw two tangents, one to the arc **A B** in its plane and one to **A C** in its plane. Through **B** and **C** prolong radii of the sphere to meet the tangents in **B′** and **C′**. The angle **B′ A C′** between the tangents is the same as the angle A of the spherical triangle in Fig. 66, where the arcs **A B** and **A C** are each less than 90°; is equal to

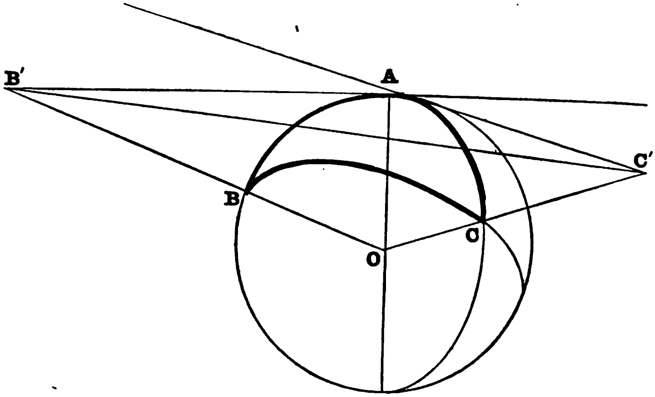

Fig. 66.

A, being vertically opposite to it, in Fig. 67, where these arcs are each greater than 90°; and is equal to the supplement of *A* in Fig. 68, where one of the arcs is less, and the other greater, than 90°. The angle B′O C′ is measured by the side *a* of the spherical triangle in Figs. 66 and 67; but in Fig. 68, the angle B′O C′ is measured by the supplement of *a*.

In the first two figures we have, by Art. 182,

$$\overline{C'B'}^2=\overline{OC'}^2+\overline{OB'}^2-2\,OC'\times OB'\cos a \quad . \; . \; . \; . \quad (i)$$
$$\overline{C'B'}^2=\overline{AC'}^2+\overline{AB'}^2-2\,AC'\times AB'\cos A \quad . \; . \; . \; . \quad (ii)$$

By subtracting (ii) from (i) and observing that

$$\overline{OC'}^2-\overline{AC'}^2=\overline{OA}^2$$
$$\overline{OB'}^2-\overline{AB'}^2=\overline{OA}^2$$

we have

$$0=2\,\overline{OA}^2+2\,AC'\times AB'\cos A-2\,OC'\times OB'\cos a$$

or, dividing through by 2 O A × O A,

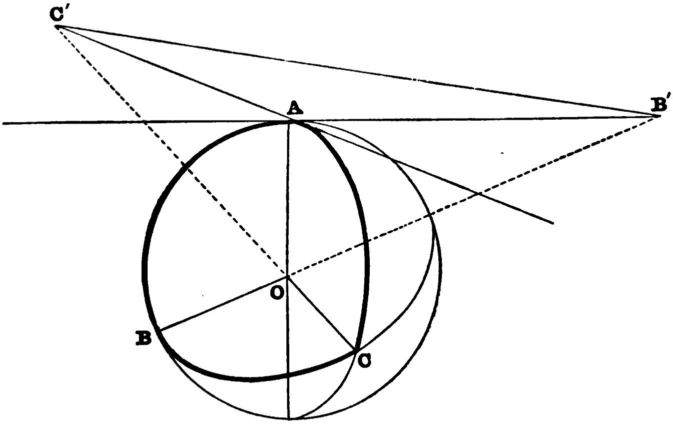

Fig. 67.

$$0=1+\frac{\mathrm{AC'}\times\mathrm{AB'}}{\mathrm{OA}\times\mathrm{OA}}\cos A-\frac{\mathrm{OC'}\times\mathrm{OB'}}{\mathrm{OA}\times\mathrm{OA}}\cos a \quad . \quad . \quad \text{(iii)}$$

But in the right triangles O A C′ and O A B′, by definition, in Fig. 66,

$$\frac{\mathrm{AC'}}{\mathrm{OA}}=\tan b=\frac{\sin b}{\cos b}, \qquad \frac{\mathrm{AB'}}{\mathrm{OA}}=\tan c=\frac{\sin c}{\cos c},$$

$$\frac{\mathrm{OC'}}{\mathrm{OA}}=\sec b=\frac{1}{\cos b}, \qquad \frac{\mathrm{OB'}}{\mathrm{OA}}=\sec c=\frac{1}{\cos c};$$

or, in Fig. 67,

$$\frac{\mathrm{AC'}}{\mathrm{OA}}=\tan\ (180°-b)=\frac{\sin\ (180°-b)}{\cos\ (180°-b)}=\frac{\sin b}{-\cos b},\ \text{etc.};$$

which substituted in (iii) give

$$0=1+\frac{\sin b\sin c\cos A}{\cos b\cos c}-\frac{\cos a}{\cos b\cos c}$$

whence

$$\cos a=\cos b\cos c+\sin a\sin b\cos A.$$

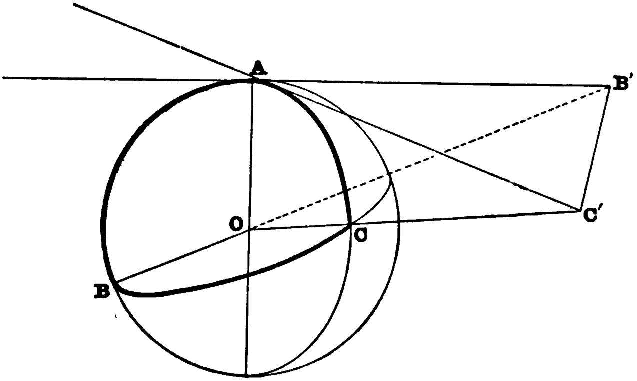

Fig. 68.

In Fig. 68 the reasoning is the same, but instead of A and a their supplements $180° - A$ and $180° - a$ are used, so that the equation (iii) has the form

$$0 = 1 + \frac{\mathrm{AC'} \times \mathrm{AB'}}{\mathrm{OA} \times \mathrm{OA}} \cos(180° - A) - \frac{\mathrm{OC'} \times \mathrm{OB'}}{\mathrm{OA} \times \mathrm{OA}} \cos(180° - a) \quad \text{(iv)}$$

But by definition,

$$\frac{\mathrm{AC'}}{\mathrm{OA}} = \tan b = \frac{\sin b}{\cos b}, \quad \frac{\mathrm{AB'}}{\mathrm{OA}} = \tan(180° - c) = \frac{\sin(180° - c)}{\cos(180° - c)},$$

$$\frac{\mathrm{OC'}}{\mathrm{OA}} = \sec b = \frac{1}{\cos b}, \quad \frac{\mathrm{OB'}}{\mathrm{OA}} = \sec(180° - c) = \frac{1}{\cos(180° - c)},$$

which substituted in (iv) give

$$0 = 1 + \frac{\sin b \sin(180° - c)}{\cos b \cos(180° - c)} \cos(180° - A) - \frac{\cos(180° - a)}{\cos b \cos(180° - c)}$$

whence, as before,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

By a cyclic change of letters (Art. 173), we find two more equations, which, with the one just proved, completely express one set of relations between the parts of a spherical triangle, thus:

$$\left.\begin{array}{l}\cos a=\cos b\cos c+\sin b\sin c\cos A\\ \cos b=\cos c\cos a+\sin c\sin a\cos B\\ \cos c=\cos a\cos b+\sin a\sin b\cos C\end{array}\right\}\quad\cdot\ \cdot\ \cdot\quad[152]$$

These are the fundamental equations of spherical trigonometry. From them are or can be derived all other general equations relating to spherical triangles.

226. *The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*

The equations expressing this theorem are thus deduced from [152]:

From the first and second of those equations, by addition and subtraction, we have

$$\cos a+\cos b=(\cos b+\cos a)\cos c+(\sin b\cos A+\sin a\cos B)\sin c$$
$$\cos a-\cos b=(\cos b-\cos a)\cos c+(\sin b\cos A-\sin a\cos B)\sin c$$

whence, by uniting terms and factoring,

$$(\cos a+\cos b)\ (1-\cos c)=(\sin b\cos A+\sin a\cos B)\sin c$$
$$(\cos a-\cos b)\ (1+\cos c)=(\sin b\cos A-\sin a\cos B)\sin c$$

The product of these two equations is

$$(\cos^2 a-\cos^2 b)\ (1-\cos^2 c)=(\sin^2 b\cos^2 A-\sin^2 a\cos^2 B)\sin^2 c$$

or, dividing by the equal factors $1-\cos^2 c$ and $\sin^2 c$,

$$\cos^2 a-\cos^2 b=\sin^2 b\cos^2 A-\sin^2 a\cos^2 B$$

By replacing in this every $\cos^2$ by its equivalent $1-\sin^2$ we get

$$1-\sin^2 a-1+\sin^2 b=\sin^2 b\,(1-\sin^2 A)-\sin^2 a\,(1-\sin^2 B)$$

which reduces to

$$0= -\sin^2 b \sin^2 A+\sin^2 a \sin^2 B$$

or

$$\frac{\sin^2 a}{\sin^2 A}=\frac{\sin^2 b}{\sin^2 B}$$

In the same manner can be deduced, from the second and third equations of [152], the equation

$$\frac{\sin^2 b}{\sin^2 B}=\frac{\sin^2 c}{\sin^2 C}$$

Combining these results, and taking the square roots, we have

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}=M \quad . \quad . \quad . \quad [153]$$

which expresses the theorem that was to be proved.

The double sign ± for these square roots is not necessary; because, when sides and angles are limited not to exceed 180°, the terms of each ratio and the ratios themselves are all positive, and for all other forms of the triangle, as will be shown further on, the three ratios become positive or negative simultaneously.

227. The common value of the three ratios is called the *modulus* of the spherical triangle. It is denoted by M. The modulus is positive when the sides and angles are limited not to exceed 180°; but, when this limitation is removed, the modulus of some triangles is positive and of others negative.

POLES AND POLAR RADII.

228. The diameter of a sphere which is perpendicular to the plane of a circle is called *the axis* of that circle, and the ends of the axis are *the poles* of the circle. This applies to small as well as to great circles. For example, the north and south poles of the earth, regarded as a sphere, are the poles not only of the equator, a great circle, but also of the parallels of latitude, which are small circles.

229. An arc of a great circle, drawn from the pole of a circle to any point of its circumference, is called the *polar radius* of the circle. The polar radius of a great circle is an arc of 90°, and of a small circle an arc of less than 90° or more than 90°. For example, the polar radius of the equator is a quadrant of the meridian; and, from the north pole, the polar radius of any parallel of north latitude is less, and that of any parallel of south latitude is greater, than a quadrant.

POLAR TRIANGLES.

230. Let A′, B′, and C′ be the poles of the sides a, b, and c of a given spherical triangle, so taken that the pole A′ and the vertex A of the triangle are on the same side of the plane of a, the pole B′ and the vertex B on the same side of the plane of b, and the pole C′ and the vertex C on the same side of the plane of c. Then the three arcs of great circles joining the points A′, B′, and C form a spherical triangle, which is called the *polar triangle* in relation to the given triangle.

231. Let the sides of this polar triangle be denoted by a', b', and c'. It is proved in geometry that, as the points A′, B′ and C′ are respectively the poles of a, b, and c, so the points A, B, and C (vertices of the original triangle) are respectively the poles of a', b', and c'; that is, if one spherical triangle is polar to another, then, reciprocally, the second triangle is polar to the first.

The relations between the parts of a spherical triangle and those of its polar are thus expressed:

$$\left.\begin{array}{ll} a+A'=180^\circ & A+a'=180^\circ \\ b+B'=180^\circ & B+b'=180^\circ \\ c+C'=180^\circ & C+c'=180^\circ \end{array}\right\} \quad . \quad . \quad [154]$$

where accented letters denote parts of the polar triangle.

232. The six quantities

$$a',\ b',\ c',\ A',\ B',\ C',$$

being the sides and angles of a spherical triangle, must satisfy the fundamental equations [**152**]; therefore their six equivalents taken from [**154**], namely,

$$180^\circ-A,\ 180^\circ-B,\ 180^\circ-C,\ 180^\circ-a,\ 180^\circ-b,\ 180^\circ-c,$$

satisfy the same equations.

The substitution of these last in the first equation of [**152**] gives

$$\begin{aligned} \cos(180^\circ-A) &= \cos(180^\circ-B)\cos(180^\circ-C) \\ &\quad +\sin(180^\circ-B)\sin(180^\circ-C)\cos(180^\circ-a) \end{aligned}$$

which reduces to the first equation of the following set:

$$\left.\begin{array}{l} \cos A=-\cos B\cos C+\sin B\sin C\cos a \\ \cos B=-\cos C\cos A+\sin C\sin A\cos b \\ \cos C=-\cos A\cos B+\sin A\sin B\cos c \end{array}\right\}. \quad [155]$$

the second and third being obtained in like manner.

The equations [**155**] express new relations between the parts of the original triangle. To describe the method of reasoning here used in finding these new relations, we say that equations [**155**] are obtained by applying [**152**] to the polar triangle. Are any new relations to be found by applying [**153**] to the polar triangle?

233. But it is worth while to deduce [155] from [152] by a purely algebraic process, having no reference to the polar triangle nor to geometric considerations of any kind. The process is by elimination. First, eliminate b and c from the first and second equations of [152], and the result reduced is the first equation of [155]; next eliminate c and a from the second and third equations of [152] to get the second equation of [155]; and lastly eliminate a and b from the third and first equations of [152] to get the third equation of [155]. It will be enough to give one of these eliminations. In the first and second equations of [152] substitute for cos c its equivalent, the second member of the third equation, whereby these equations become

$$\cos a = \cos b\,(\cos a \cos b + \sin a \sin b \cos C) + \sin b \sin c \cos A$$
$$\cos b = \cos a\,(\cos a \cos b + \sin a \sin b \cos C) + \sin c \sin a \cos B$$

which reduce to

$$\cos a\,(1 - \cos^2 b) = \cos b \sin a \sin b \cos C + \sin b \sin c \cos A$$
$$\cos b\,(1 - \cos^2 a) = \cos a \sin a \sin b \cos C + \sin c \sin a \cos B$$

or, by dividing the first through by sin b, and the second through by sin a, observing that $1 - \cos^2 b = \sin^2 b$ and $1 - \cos^2 a = \sin^2 a$,

$$\cos a \sin b = \cos b \sin a \cos C + \sin c \cos A$$
$$\cos b \sin a = \cos a \sin b \cos C + \sin c \cos B$$

Now substitute in these equations for

$$\sin a, \qquad \sin b, \qquad \text{and } \sin c,$$

their equivalents,

$$M \sin A, \qquad M \sin B, \qquad \text{and } M \sin C,$$

taken from [153]. The results, divided through by the common factor M, are

$$\cos a \sin B = \cos b \sin A \cos C + \sin C \cos A$$
$$\cos b \sin A = \cos a \sin B \cos C + \sin C \cos B$$

from which it only remains to eliminate cos b. This is done by equating the two values of cos b taken from the two equations, thus,

$$\frac{\cos a \sin B - \sin C \cos A}{\sin A \cos C} = \frac{\cos a \sin B \cos C + \sin C \cos B}{\sin A}$$

and this equation reduces to

$$\cos a \sin B - \sin C \cos A = \cos a \sin B \cos^2 C + \sin C \cos B \cos C$$

or

$$\cos a \sin B\ (1 - \cos^2 C) - \sin C \cos A = \sin C \cos B \cos C$$

or

$$\cos a \sin B \sin C - \cos A = \cos B \cos C$$

or

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

the first equation of [155].

The working out of the other two eliminations is suggested for an exercise.

RELATIONS BETWEEN ANY FIVE PARTS.

234. The fundamental equations [152] express relations between an angle and the three sides of a spherical triangle; and the equations [155] derived from the fundamental equations through the polar triangle express relations between a side and the three angles. Four parts enter into each equation. Equations [153] also express relations between four parts, two sides and the opposite angles. We shall now develop the relations between any five parts, the three sides and two angles or the three angles and two sides.

235. Let the first and second equations of [152] be combined by substituting in the first, for cos b, the right-hand member of the second, which gives

$$\cos a = (\cos c \cos a + \sin c \sin a \cos B) \cos c + \sin b \sin c \cos A$$

or

$$\cos a = \cos^2 c \cos a + \cos c \sin c \sin a \cos B + \sin b \sin c \cos A$$

or

$$\cos a\,(1 - \cos^2 c) = \cos c \sin c \sin a \cos B + \sin b \sin c \cos A$$

or

$$\cos a \sin c = \cos c \sin a \cos B + \sin b \cos A$$

one equation of the kind required.

Or, combine the same two equations by substituting in the second, for cos a, the right-hand member of the first, which gives, after reductions similar to the above,

$$\cos b \sin c = \cos c \sin b \cos A + \sin a \cos B$$

In the same way, the second and third equations of [152] give two equations, and the third and first give two more. These six equations are

$$\left.\begin{aligned}
\cos a \sin c &= \cos c \sin a \cos B + \sin b \cos A \\
\cos b \sin c &= \cos c \sin b \cos A + \sin a \cos B \\
\cos b \sin a &= \cos a \sin b \cos C + \sin c \cos B \\
\cos c \sin a &= \cos a \sin c \cos B + \sin b \cos C \\
\cos c \sin b &= \cos b \sin c \cos A + \sin a \cos C \\
\cos a \sin b &= \cos b \sin a \cos C + \sin c \cos A
\end{aligned}\right\} . \quad [156]$$

236. In like manner may be derived from [155] the following set of six equations:

$$\left.\begin{aligned}\cos A\sin C&=-\cos C\sin A\cos b+\sin B\cos a\\ \cos B\sin C&=-\cos C\sin B\cos a+\sin A\cos b\\ \cos B\sin A&=-\cos A\sin B\cos c+\sin C\cos b\\ \cos C\sin A&=-\cos A\sin C\cos b+\sin B\cos c\\ \cos C\sin B&=-\cos B\sin C\cos a+\sin A\cos c\\ \cos A\sin B&=-\cos B\sin A\cos c+\sin C\cos a\end{aligned}\right\}.\,[157]$$

The equations [157] may be derived from [156] directly, by applying the latter to the polar triangle.

RELATIONS BETWEEN ANY FOUR CONSECUTIVE PARTS.

237. By four consecutive parts of a spherical triangle is meant any four parts that follow one another in passing around the triangle, as *A*, *c*, *B*, *a*, or *b*, *C*, *a*, *B*. A set of six equations expressing relations between any four consecutive parts is derived from [156] by eliminating from each equation one of the sides, thus:

Divide the first equation of [156] through by sin *a*, the second by sin *b*, the third by sin *b*, the fourth by sin *c*, the fifth by sin *c*, and the sixth by sin *a*. Replace the ratios,

$$\frac{\sin b}{\sin a},\quad \frac{\sin a}{\sin b},\ \text{etc.},$$

thus produced in the right-hand members by their respective equivalents,

$$\frac{\sin B}{\sin A},\quad \frac{\sin A}{\sin B},\ \text{etc.}$$

taken from [153], the results reduced are

$$\left.\begin{aligned}
\cot a \sin c &= \cos c \cos B + \sin B \cot A \\
\cot b \sin c &= \cos c \cos A + \sin A \cot B \\
\cot b \sin a &= \cos a \cos C + \sin C \cot B \\
\cot c \sin a &= \cos a \cos B + \sin B \cot C \\
\cot c \sin b &= \cos b \cos A + \sin A \cot C \\
\cot a \sin b &= \cos b \cos C + \sin C \cot A
\end{aligned}\right\} \quad . \quad [158]$$

238. If these equations be applied to the polar triangle, each equation will change into another of this same set, and no new relations are developed; or, if we eliminate from each equation of [157] one of the angles, the result is [158] again, and no new relations are developed.

TRANSFORMATIONS OF THE GENERAL EQUATIONS.

239. The foregoing general equations are sufficient for the solution of all spherical triangles in which any three parts are given to find the other three; but with the exception of [153] they are not in convenient form for logarithmic computation. It is the object of the following transformations to make them so.

240. In the first equation of [152] let $\cos A$ be replaced by its equivalent $1-2\sin^2 \frac{1}{2} A$, taken from [84]. The result, reduced by [64] is

$$\cos a = \cos (b-c) - 2 \sin b \sin c \sin^2 \tfrac{1}{2} A \quad . \quad [159]$$

From this equation, by division, we have

$$\sin^2 \tfrac{1}{2} A = \frac{\cos (b-c) - \cos a}{2 \sin b \sin c}$$

the numerator of the second member of which is by [95] changed to the form

$$2 \sin \tfrac{1}{2} (a+b-c) \sin \tfrac{1}{2} (a-b+c)$$

so that

$$\sin^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(a-b+c) \sin \frac{1}{2}(a+b-c)}{\sin b \sin c}$$

In the same way, from the second and third equations of [152]

$$\sin^2 \tfrac{1}{2} B = \frac{\sin \frac{1}{2}(a+b-c) \sin \frac{1}{2}(-a+b+c)}{\sin c \sin a}$$

$$\sin^2 \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(-a+b+c) \sin \frac{1}{2}(a-b+c)}{\sin a \sin b}$$

These equations are now in a form convenient for logarithmic computation, but they are usually shortened by introducing a symbol s, which is thus defined:

whence

$$\left.\begin{aligned} s &= \tfrac{1}{2}(a+b+c) \\ s-a &= \tfrac{1}{2}(-a+b+c) \\ s-b &= \tfrac{1}{2}(a-b+c) \\ s-c &= \tfrac{1}{2}(a+b-c) \end{aligned}\right\} \quad . \ . \ . \quad [160]$$

By substituting these equivalents and taking the square roots, the equations become

$$\left.\begin{aligned} \sin \tfrac{1}{2} A &= \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}} \\ \sin \tfrac{1}{2} B &= \sqrt{\frac{\sin (s-c) \sin (s-a)}{\sin c \sin a}} \\ \sin \tfrac{1}{2} C &= \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin a \sin b}} \end{aligned}\right\} \quad . \ . \quad [161]$$

241. Again, let cos A in the first equation of [152] be replaced by its equivalent $-1 + 2\cos^2 \frac{1}{2} A$, taken from [83]. The result by [62] is

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NOTE.—The double sign ± before the radicals may be neglected when the formulas are applied to triangles whose sides and angles do not exceed 180°; otherwise it should be taken into account.

$$\cos a = \cos(b+c) + 2 \sin b \sin c \cos^2 \tfrac{1}{2} A \quad . \quad . \quad . \qquad [162]$$

whence

$$\cos^2 \tfrac{1}{2} A = \frac{\cos a - \cos(b+c)}{2 \sin b \sin c}.$$

By [95] the numerator of the right-hand member of this is changed to the form

$$2 \sin \tfrac{1}{2}(a+b+c) \sin \tfrac{1}{2}(-a+b+c)$$

so that

$$\cos^2 \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2}(a+b+c) \sin \tfrac{1}{2}(-a+b+c)}{\sin b \sin c}$$

Also in the same way

$$\cos^2 \tfrac{1}{2} B = \frac{\sin \tfrac{1}{2}(a+b+c) \sin \tfrac{1}{2}(a-b+c)}{\sin c \sin a}$$

$$\cos^2 \tfrac{1}{2} C = \frac{\sin \tfrac{1}{2}(a+b+c) \sin \tfrac{1}{2}(a+b-c)}{\sin a \sin b}$$

By substituting from [160] and taking the square roots these equations become

$$\left.\begin{aligned} \cos \tfrac{1}{2} A &= \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \\ \cos \tfrac{1}{2} B &= \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}} \\ \cos \tfrac{1}{2} C &= \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad [163]$$

242. If each equation of [161] be divided by the corresponding one of [163], the results are

$$\left.\begin{aligned} \tan \tfrac{1}{2} A &= \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin s \sin(s-a)}} \\ \tan \tfrac{1}{2} B &= \sqrt{\frac{\sin(s-c) \sin(s-a)}{\sin s \sin(s-b)}} \\ \tan \tfrac{1}{2} C &= \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin s \sin(s-c)}} \end{aligned}\right\} \quad . \quad . \qquad [164]$$

243. From [155], by steps precisely similar to the foregoing, and putting for shortness

$$\left.\begin{aligned} S&=\tfrac{1}{2}(A+B+C) \\ S-A&=\tfrac{1}{2}(-A+B+C) \\ S-B&=\tfrac{1}{2}(A-B+C) \\ S-C&=\tfrac{1}{2}(A+B-C) \end{aligned}\right\} \quad . \; . \; . \; . \quad [165]$$

may be derived sets of formulas for the sines, cosines, and cotangents of the halves of the sides, as follows:

$$\cos A = -\cos(B+C) - 2\sin B\sin C\sin^2\tfrac{1}{2}a \quad . \; . \quad [166]$$

$$\left.\begin{aligned} \sin\tfrac{1}{2}a &= \sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}} \\ \sin\tfrac{1}{2}b &= \sqrt{\frac{-\cos S\cos(S-B)}{\sin C\sin A}} \\ \sin\tfrac{1}{2}c &= \sqrt{\frac{-\cos S\cos(S-C)}{\sin A\sin B}} \end{aligned}\right\} \quad . \; . \; . \; . \quad [167]$$

$$\cos A = -\cos(B-C) + 2\sin B\sin C\cos^2\tfrac{1}{2}a \quad . \; . \quad [168]$$

$$\left.\begin{aligned} \cos\tfrac{1}{2}a &= \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}} \\ \cos\tfrac{1}{2}b &= \sqrt{\frac{\cos(S-C)\cos(S-A)}{\sin C\sin A}} \\ \cos\tfrac{1}{2}c &= \sqrt{\frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}} \end{aligned}\right\} \quad . \; . \; . \quad [169]$$

$$\left.\begin{aligned} \cot\tfrac{1}{2}a &= \sqrt{\frac{\cos(S-B)\cos(S-C)}{-\cos S\cos(S-A)}} \\ \cot\tfrac{1}{2}b &= \sqrt{\frac{\cos(S-C)\cos(S-A)}{-\cos S\cos(S-B)}} \\ \cot\tfrac{1}{2}c &= \sqrt{\frac{\cos(S-A)\cos(S-B)}{-\cos S\cos(S-C)}} \end{aligned}\right\} \quad . \; . \; . \quad [170]$$

[illegible]

[illegible]

[illegible] sum and of the dif- [illegible] [64].

[illegible]

[illegible]

which is the first of Gauss's Equations. The second, third, and fourth are obtained by applying like reductions to the other three equations of the four above given.

$$\left.\begin{aligned} \frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}C} &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c} \\ \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C} &= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c} \\ \frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C} &= \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c} \\ \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C} &= \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c} \end{aligned}\right\} \quad . \; . \; . \quad [171]$$

NAPIER'S ANALOGIES.

245. Equations known as Napier's Analogies may be obtained from Gauss's Equations by dividing the latter one by another. They are as follows:

$$\left.\begin{aligned} \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} &= \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)} \\ \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} &= \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)} \\ \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)} &= \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)} \\ \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)} &= \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A+B)} \end{aligned}\right\} \quad . \; . \; . \quad [172]$$

SPECIAL FORMULAS FOR SPHERICAL RIGHT-ANGLED TRIANGLES.

246. When one of the angles of a spherical triangle is a right angle, some of the general formulas assume simple forms, which for convenience are here collected for reference. When, for example, the angle C is 90°, and consequently $\cos C=0$, the third equation of [152] becomes

$$\cos c=\cos a \cos b$$

If $C=90°$, and consequently $\sin C=1$, equations [153] become

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\sin c$$

whence

$$\begin{aligned}\sin a&=\sin c \sin A\\ \sin b&=\sin c \sin B\end{aligned}$$

If $C=90°$, equations [155] become

$$\begin{aligned}&\cos A=\sin B \cos a\\ &\cos B=\sin A \cos b\\ &0=-\cos A \cos B+\sin A \sin B \cos c\end{aligned}$$

the last of which, by dividing through by $\sin A \sin B$ becomes

$$\cos c=\cot A \cot B.$$

If $C=90°$, the last four equations of [158] become, after a little reduction,

$$\begin{aligned}\sin a&=\tan b \cot B\\ \cos B&=\cot c \tan a\\ \cos A&=\cot c \tan b\\ \sin b&=\tan a \cot A\end{aligned}$$

247. Thus we have found ten simple equations by means of which any part of a spherical right triangle can be computed when two of the other parts are given. Using h instead of c to denote the side opposite the right angle, the equations are thus written in a convenient order for reference:

$$\left.\begin{array}{lr}
\sin a = \sin h \sin A & \ldots 1 \\
\sin a = \tan b \cot B & \ldots 2 \\
\sin b = \sin h \sin B & \ldots 3 \\
\sin b = \tan a \cot A & \ldots 4 \\
\cos A = \cos a \sin B & \ldots 5 \\
\cos A = \cot h \tan b & \ldots 6 \\
\cos B = \cos b \sin A & \ldots 7 \\
\cos B = \cot h \tan a & \ldots 8 \\
\cos h = \cos a \cos b & \ldots 9 \\
\cos h = \cot A \cot B & \ldots 10
\end{array}\right\} \quad \ldots \quad [173]$$

NAPIER'S RULES.

248. These ten equations [173] may be kept in memory by means of two simple rules based upon an ingenious device first suggested by Baron Napier, the inventor of logarithms. The device is to take as *parts* the two legs a and b of the right triangle, the complements of the two oblique angles, that is $90° - A$ and $90° - B$, and the complement of the hypothenuse, $90° - h$; and to regard these five parts as forming a ring around the triangle. The right angle is not taken into account. In this ring, any part may be taken as *middle part;* and then the two parts next to it, one on the right hand and one on the left, are called the *adjacent parts,* while the other two parts, not being next to it, are called the *opposite parts.* For example, if a is taken as middle part, co B and b are the adjacent parts, while co h and co A are the opposite parts; or if co A is taken as middle part, b and co h are

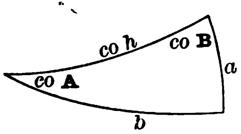

Fig. 69.

the adjacent parts, while a and co B are the opposite parts.

These are Napier's rules:

(i) *The sine of a middle part is equal to the product of the tangents of the adjacent parts.*

(ii) *The sine of a middle part is equal to the product of the cosines of the opposite parts.*[*]

To prove these rules, let them be applied to the spherical right triangle (Fig. 69), each of the five parts in turn being made the middle part. The results will be the ten equations [173]. For example, let a be the middle part. By the first rule

$$\sin a = \tan (\text{co } B) \tan b$$

or

$$\sin a = \cot B \tan b$$

which is the second equation of [173].

By the second rule

$$\sin a = \cos (\text{co } h) \cos (\text{co } A)$$

or

$$\sin a = \sin h \sin A$$

which is the first equation of [173].

249. The general formulas for spherical triangles likewise assume simple forms when one of the sides is a quadrant. Such triangles are called *quadrantal* triangles; but they do not require special treatment.

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[*] The similarity in respect to vowels between the words *tangents* and *adjacent*, and between the words *cosines* and *opposite*, may aid in remembering these rules.

INSCRIBED AND CIRCUMSCRIBED SMALL CIRCLES.

250. The geometrical meaning of some of the formulas of this chapter is analogous to that of the corresponding formulas for plane triangles.

For example, let us put for abbreviation

$$k=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

Then the equations [164] become

$$\tan\tfrac{1}{2}A=\frac{k}{\sin(s-a)},\ \text{etc.}$$

Now suppose a spherical triangle to be drawn with a small circle of polar radius r inscribed in it and touching its sides at D, E, and F. We easily see that the arcs

$$\begin{aligned} \text{AD}&=\text{AF}=s-a\\ \text{BE}&=\text{BD}=s-b\\ \text{CF}&=\text{CE}=s-c \end{aligned}$$

and that the angles A, B, and C are bisected by the arcs O A, O B, and O C, drawn from O, the pole of the inscribed circle, to the angles of the triangle. Applying Napier's first rule to any of these right spherical triangles, as A O D, taking

$$\begin{array}{lll} \text{arc} & \text{AD} & =s-a \text{ as middle part} \end{array}$$

$$\left.\begin{array}{lll} \text{co angle} & \text{OAD} & =\text{co }\tfrac{1}{2}A \\ \text{arc} & \text{OD} & =r \end{array}\right\}\ \text{as } \textit{adjacent} \text{ parts}$$

we have

$$\sin(s-a)=\cot\tfrac{1}{2}A\tan r$$

whence

$$\tan \tfrac{1}{2} A = \frac{\tan r}{\sin (s-a)}.$$

Comparing this with

$$\tan \tfrac{1}{2} A = \frac{k}{\sin (s-a)}$$

we discover that

$$\tan r = k$$

or

$$\tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}} \quad . \quad . \quad [174]$$

By means of this relation the equations [164] take the following symmetrical form:

$$\left.\begin{aligned} \tan \tfrac{1}{2} A &= \frac{\tan r}{\sin (s-a)} \\ \tan \tfrac{1}{2} B &= \frac{\tan r}{\sin (s-b)} \\ \tan \tfrac{1}{2} C &= \frac{\tan r}{\sin (s-c)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad [175]$$

251. If we put for abbreviation

$$K = \sqrt{\frac{\cos (S-A) \cos (S-B) \cos (S-C)}{-\cos S}}$$

the equations [170] become

$$\cot \tfrac{1}{2} a = \frac{K}{\cos (S-A)}, \text{ etc.}$$

Now let a small circle be circumscribed about the spherical triangle, and denote its polar radius by *R*. The arcs O A, O B, and O C drawn from the pole of this circle to the vertices of the triangle are each equal to *R*;

and they divide the triangle into three isosceles triangles. Each of the isosceles triangles is divided into two equal right triangles by the arcs **O P**, **O Q**, and **O R** drawn from the pole of the circle perpendicular to the sides *a*, *b*, and *c* respectively. The sides *a*, *b*, and *c* are bisected at the points **P**, **Q**, and **R**. It is easy to prove that the angles

$$\begin{aligned} \mathrm{PCO} &= \mathrm{PBO} = S-A \\ \mathrm{QAO} &= \mathrm{QCO} = S-B \\ \mathrm{RBO} &= \mathrm{RAO} = S-C \end{aligned}$$

Applying Napier's first rule to any of these right spherical triangles, as **P C O**, taking

$$\text{co. angle } \mathrm{PCO} = \text{co. } (S-A) \text{ as } \textit{middle} \text{ part}$$

$$\left.\begin{aligned} \text{arc } \quad \mathrm{CP} &= \tfrac{1}{2}\,a \\ \text{co. arc } \quad \mathrm{OC} &= \text{co. } R \end{aligned}\right\} \text{ as } \textit{adjacent} \text{ parts.}$$

we have

$$\cos(S-A) = \tan \tfrac{1}{2}\,a \cot R$$

whence

$$\cot \tfrac{1}{2}\,a = \frac{\cot R}{\cos(S-A)}$$

Comparing this with

$$\cot \tfrac{1}{2}\,a = \frac{K}{\cos(S-A)}$$

we discover that

$$\cot R = K$$

or

$$\cot R = \sqrt{\frac{\cos(S-A)\cos(S-B)\cos(S-C)}{-\cos S}} \quad . \quad . \quad [176]$$

By means of this relation the equations [170] take the following symmetrical form:

$$\left.\begin{aligned}\cot \tfrac{1}{2} a &= \frac{\cot R}{\cos (S-A)}\\ \cot \tfrac{1}{2} b &= \frac{\cot R}{\cos (S-B)}\\ \cot \tfrac{1}{2} c &= \frac{\cot R}{\cos (S-C)}\end{aligned}\right\} \quad . \; . \; . \; . \; . \quad [177]$$

THE MODULUS OF A SPHERICAL TRIANGLE.

252. By [161] and [163]

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}$$

and by [77],

$$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A$$

Therefore, by substitution in the latter,

$$\sin A = \frac{2\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}}{\sin b \sin c}$$

whence

$$\sin A \sin b \sin c = 2\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

But by [174]

$$\tan r \sin s = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}$$

Therefore

$$\sin A \sin b \sin c = 2 \tan r \sin s \quad . \; . \; . \quad [178]$$

Again by [167] and [169],

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S-A)}{\sin B \sin C}}$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}}$$

and, by [77],

$$\sin a = 2 \sin \tfrac{1}{2} a \cos \tfrac{1}{2} a$$

Therefore, by substitution in the latter,

$$\sin a = \frac{2\sqrt{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)}}{\sin B \sin C}$$

whence

$$\sin a \sin B \sin C = 2\sqrt{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)}$$

But by [176]

$$\cot R \cos S = \sqrt{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)}.$$

Therefore

$$\sin a \sin B \sin C = 2 \cot R \cos S \quad . \quad . \quad [179]$$

Dividing [178] by [179], member by member, gives

$$\frac{\sin A \sin b \sin c}{\sin a \sin B \sin C} = \frac{\tan r \sin s}{\cot R \cos S}$$

or, since by [153] $\frac{\sin A \sin b}{\sin a \sin B} = 1$,

$$\frac{\sin c}{\sin C} = \frac{\tan r \sin s}{\cot R \cos S}$$

Therefore, M denoting the modulus (Art. 227),

$$M = \frac{\tan r \sin s}{\cot R \cos S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [180]$$

CHAPTER IX.

THE SOLUTION OF SPHERICAL RIGHT TRIANGLES.

253. The given parts of a spherical right triangle, besides the right angle, may be:

I. An oblique angle and the hypothenuse, as A and h.
II. An oblique angle and the opposite leg, as A and a.
III. An oblique angle and the adjacent leg, as A and b.
IV. The hypothenuse and one leg, as h and a.
V. The two legs a and b.
VI. The two oblique angles A and B.

254. The formulas for solution in each case are selected from [173], or they are remembered by means of Napier's Rules. Each formula should connect one required part with two given parts. When the functions of the three required parts have been computed, the work may be tested by means of an equation selected from [173]. One of these computed functions is always equal to the product of the other two; so that, if logarithms are used, one logarithm is always equal to the sum of the other two.

255. In the examples that follow, the given parts do not exceed the limit 180°, but results exceeding this limit are not rejected. On the contrary, they are retained and interpreted; and thus it is shown that the problem always has two solutions. Each computed part has two values between 0° and 360°; and the question to be settled is, which of these values belong together as the parts of one triangle, and which as parts of the other. The answer comes from two equations, selected from [173], or given by Napier's rules, each of which presents a function of a given part as equal to the product or to the quotient of functions of two required parts. The positive or negative character of this product or quotient being thus known, it is only necessary to select the computed values of the required parts in such quadrants as to make the algebraic signs of their functions alike or unlike accordingly. This will be illustrated in detail under each case.

I.

256. In a spherical right triangle, given A and h, to find a, b, and B.

By [173] or Napier's rules,

For solution,

$$\sin a = \sin A \sin h \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$\tan b = \cos A \tan h \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

$$\cot B = \tan A \cos h \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

$$a_2 = 180° - a_1$$

$$b_2 = 180° + b_1$$

$$B_2 = 180° + B_1$$

For a test,

$$\sin a = \tan b \cot B \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

For discriminating between values,

$$\sin b = \frac{\tan a}{\tan A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

$$\sin B = \frac{\cos A}{\cos a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

257. The first equation gives two values of a, each less than 180°; for, since A and h are limited to the first two quadrants, both $\sin A$ and $\sin h$ are positive, making $\sin a$ positive, thus limiting a to the first two quadrants. These two values of a, having the same sine, are supplements of each other. The second equation gives two values of b

which differ by 180°; for $\tan b = \tan (b + 180°)$. Also the third equation gives two values of B which differ by 180°. By the fifth and sixth equations, we see that when a is taken in the same quadrant (first or second) with the given A, making the algebraic signs of $\tan a$ and $\tan A$ also of $\cos a$ and $\cos A$ alike, then $\sin b$ and $\sin B$ are both positive, so that b and B must be taken in the first or second quadrant; but when a is taken not in the same quadrant with the given A, making the algebraic signs of $\tan a$ and $\tan A$ also of $\cos a$ and $\cos A$ unlike, then $\sin b$ and $\sin B$ are both negative, so that b and B must be taken in the third or fourth quadrant. One triangle, therefore, contains that value of a which is in the same quadrant with the given A and the values of b and B which do not exceed 180°, while the other triangle contains the other values.

258. The geometrical solution of the problem gives the same results. It is made on the surface of a globe as follows: Draw two great-circle arcs forming an angle equal to the given value of A; on one side of this angle measure off A B equal to the given value of h, but extend the other side, b, to form a complete circumference; perpendicular to b and passing through the point B, draw a semi-circumference meeting b at two diametrically opposite points C_1 and C_2. Thus are formed two spherical right triangles AC_1B and AC_2B, each containing with the given parts a different value of each required part. The two values of a are the arcs C_1B and C_2B, which together form a semi-circumference. The two values of b are the arcs AC_1 and AC_2, the latter exceeding the former by a semi-circumference. The two values of B are ABC_1 and ABC_2, the latter being a convex angle exceeding the former by 180°.

259. Examples.

1. Given $A = 110^\circ\ 39'$, and $h = 63^\circ\ 29'$.

By three-place natural functions. (See Table I, page 9.)

(i) $\sin a = 0.936 \times 0.895 = 0.838$, $a = 56^\circ\ 54'$ or $123^\circ\ 6'$
(ii) $\tan b = -0.352 \times 2.005 = -0.706$, $b = 144^\circ\ 47'$ or $324^\circ\ 47'$
(iii) $\cot B = -2.655 \times 0.446 = -1.184$, $B = 139^\circ\ 49'$ or $319^\circ\ 49'$

The value of a which is in the same quadrant with the given A is $123^\circ\ 6'$; so that

$$\begin{aligned} a &= 123^\circ\ 6' \\ b &= 144^\circ\ 47' \\ B &= 139^\circ\ 49' \end{aligned}$$

are the required parts of one triangle, and

$$\begin{aligned} a &= 56^\circ\ 54' \\ b &= 324^\circ\ 47' \\ B &= 319^\circ\ 49' \end{aligned}$$

are the required parts of the other triangle.

By five-place logarithmic functions. (See *Log. Trig. Tab.*)

	(i)	(ii)	(iii)
$A = 110^\circ\ 39'$	log sin 9.97116	log cos 9.54735 n	log tan 10.42381 n
$h = 63^\circ\ 29'$	log sin 9.95173	log tan 10.30195	log cos 9.64978
	log sin 9.92289	log tan 9.84930 n	log cot 10.07359 n
			9.84930
			Test, 9.92289

Results,

$$\left.\begin{aligned} a &= 123^\circ\ 8'\ 30'' \\ b &= 144^\circ\ 44'\ 49'' \\ B &= 139^\circ\ 49'\ 53'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 56^\circ\ 51'\ 30'' \\ b &= 324^\circ\ 44'\ 49'' \\ B &= 319^\circ\ 49'\ 53'' \end{aligned}\right.$$

2. Given $A = 72^\circ\ 14'$, and $h = 57^\circ\ 38'$.

Results,

$$\left.\begin{aligned} a &= 53^\circ\ 32'\ 53'' \\ b &= 25^\circ\ 42'\ 30'' \\ B &= 30^\circ\ 54'\ \ 6'' \end{aligned}\right\} \text{ or } \left\{\begin{aligned} a &= 126^\circ\ 27'\ \ 7'' \\ b &= 205^\circ\ 42'\ 30'' \\ B &= 210^\circ\ 54'\ \ 6'' \end{aligned}\right.$$

3. Given $A = 59^\circ\ 32'$, and $h = 124^\circ\ 15'$.

Results,

$$\left.\begin{aligned} a &= 45^\circ\ 26'\ \ 9'' \\ b &= 143^\circ\ 19'\ 32'' \\ B &= 133^\circ\ 44'\ \ 0'' \end{aligned}\right\} \text{ or } \left\{\begin{aligned} a &= 134^\circ\ 33'\ 51'' \\ b &= 323^\circ\ 19'\ 32'' \\ B &= 313^\circ\ 44'\ \ 0'' \end{aligned}\right.$$

4. Given $A = 132^\circ\ 42'$, and $h = 128^\circ\ 24'$.

Results,

$$\left.\begin{aligned} a &= 144^\circ\ 50'\ \ 0'' \\ b &= 40^\circ\ 33'\ \ 2'' \\ B &= 56^\circ\ \ 3'\ 18'' \end{aligned}\right\} \text{ or } \left\{\begin{aligned} a &= 35^\circ\ 10'\ \ 0'' \\ b &= 220^\circ\ 33'\ \ 2'' \\ B &= 236^\circ\ \ 3'\ 18'' \end{aligned}\right.$$

II.

260. Given A and a, to find h, b, and B.

By [173] or Napier's rules,

For solution,

$$\sin h = \csc A \sin a \quad \ldots\ldots \quad \text{(i)}$$
$$\sin b = \cot A \tan a \quad \ldots\ldots \quad \text{(ii)}$$
$$\sin B = \cos A \sec a \quad \ldots\ldots \quad \text{(iii)}$$

$$\begin{aligned} h_2 &= 180^\circ - h_1 \\ b_2 &= 180^\circ - b_1, \text{ or } 540^\circ - b_1 \\ B_2 &= 180^\circ - B_1, \text{ or } 540^\circ - B_1 \end{aligned}$$

For a test,

$$\sin b = \sin B \sin h \quad \ldots\ldots \quad \text{(iv)}$$

For discriminating between values,

$$\cos b = \frac{\cos h}{\cos a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{v})$$

$$\cos B = \frac{\tan a}{\tan h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{vi})$$

261. The first equation gives two supplementary values of h, each of which is less than 180°, because, the given parts being limited ($A < 180°$ and $a < 180°$), both csc A and sin a are positive, making sin h positive, thus limiting h to the first two quadrants. The second equation gives two values of b, each of which is less than 180° when A and a are in the same quadrant, making sin b positive; but each is greater than 180° when A and a are in different quadrants, making sin b negative. The same is true of the two values given by the third equation.

By the two equations chosen to discriminate between values, we see that when h is taken in the same quadrant (first or second) with the given a, the algebraic signs of cos h and cos a, also of tan h and tan a, are alike, so that cos b and cos B are both positive, limiting b and B to the first and fourth quadrants; but when h is taken not in the same quadrant with a, the algebraic signs of cos h and cos a, also of tan h and tan a, are unlike, so that cos b and cos B are both negative, limiting b and B to the second and third quadrants. This makes the determination of b and B complete, since it is already known whether each of them is less or greater than 180°.

262. The geometrical solution of this problem is as follows: Draw two great-circle arcs b and h forming an angle equal to the given value of A; extend b to form a complete circumference, and h to form a semi-circumference meeting b again in a point diametrically opposite **A**. From the pole of b, and with a polar radius equal to the difference between the given value of a and 90°, draw a small circle; the circumference of this small circle will, in general, cut the semi-circumference h in two points $\mathbf{B}_1$ and $\mathbf{B}_2$. *Through* these points, if a is less than 90°, draw polar radii to meet b in two points $\mathbf{C}_1$ and $\mathbf{C}_2$; but, if a is greater than 90°, draw polar radii *from* $\mathbf{B}_1$ and $\mathbf{B}_2$ to and through the pole of b to meet b in two points $\mathbf{C}_1$ and $\mathbf{C}_2$. Thus, in either case, are formed two spherical right triangles $\mathbf{A\,B}_1\,\mathbf{C}_1$ and $\mathbf{A\,B}_2\,\mathbf{C}_2$, each containing with the given parts a different value of each required part. The two values of h are $\mathbf{A\,B}_1$ and $\mathbf{A\,B}_2$, which are evidently supplementary. The two values of b are $\mathbf{A\,C}_1$ and $\mathbf{A\,C}_2$, which are either both less or both greater than a semi-circumference. The two values of B are $\mathbf{A\,B}_1\,\mathbf{C}_1$ and $\mathbf{A\,B}_2\,\mathbf{C}_2$, which are either both less or both greater than two right angles.

263. If the given value of a differs from 90° less than does the given value of A, the equations for solution give $\sin h > 1$, $\sin b > 1$, and $\sin B > 1$, so that the solution is impossible. The corresponding fact in the geometrical construction is that the polar radius is too short to reach the semi-circumference h, so that the two points $\mathbf{B}_1$ and $\mathbf{B}_2$ cannot be found. When a and A differ equally from 90°, the first formula gives $\sin h = 1$, and the two solutions become one. Geometrically, in this case, the polar radius just reaches the semi-circumference h, so that the points $\mathbf{B}_1$ and $\mathbf{B}_2$ fall together in the same place. Illustrations of these special cases occur among the examples.

264. Examples.

1. Given $A = 125° 40'$, and $a = 48° 54'$.

By natural functions.

$$\begin{array}{lrl}
\text{(i)} & \sin h = 1.231 \times 0.754 = 0.928, & h = 68° \ 9' \text{ or } 111° 51' \\
\text{(ii)} & \sin b = -0.718 \times 1.146 = -0.823, & b = 235° 24' \text{ or } 304° 36' \\
\text{(iii)} & \sin B = -0.583 \times 1.521 = -0.887, & B = 242° 30' \text{ or } 297° 30'
\end{array}$$

The value of h which is in the same quadrant with a, making the cosines of b and B positive, is 68° 9′; and the values of b and B are then in the fourth quadrant. Hence the results are to be arranged thus:

$$\left.\begin{array}{l} h = 68° \ 9' \\ b = 304° 36' \\ B = 297° 30' \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} h = 111° 51' \\ b = 235° 24' \\ B = 242° 30' \end{array}\right.$$

2. Given $A = 119° 30'$, and $a = 139° 40'$.

By logarithmic functions.

	(i)	(ii)	(iii)
$A = 119° 30'$	log csc 30.06030	log cot 9.75264 n	log cos 9.69234 n
$a = 139° 40'$	log sin 9.81106	log tan 9.92894 n	log sec 0.11788 n
	log sin 9.87136	log sin 9.68158	log sin 9.81022
			9.87136
			Test, 9.68158

Results,

$$\left.\begin{array}{l} h = 131° \ 57' \ 27'' \\ b = 28° \ 42' \ 37'' \\ B = 40° \ 14' \ 20'' \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} h = 48° \ 2' \ 33'' \\ b = 151° \ 17' \ 23'' \\ B = 139° \ 45' \ 40'' \end{array}\right.$$

3. Given $A = 74° 35'$, and $a = 61° 22'$.

Results,

$$\left.\begin{array}{l} h = 65° \ 34' \ 10'' \\ b = 30° \ 20' \ 11'' \\ B = 33° \ 41' \ 35'' \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} h = 114° \ 25' \ 50'' \\ b = 149° \ 39' \ 49'' \\ B = 146° \ 18' \ 25'' \end{array}\right.$$

4. Given $A = 68° 36'$, and $a = 119° 20'$.

Results,

$$\left.\begin{array}{l} h = 110° 33' 24'' \\ b = 315° 46' 55'' \\ B = 311° 51' 21'' \end{array}\right\} \text{ or } \left\{\begin{array}{l} h = 69° 26' 36'' \\ b = 224° 13' 5'' \\ B = 228° 8' 39'' \end{array}\right.$$

5. Given $A = 65°$, and $a = 75°$.

Results,

$\sin h > 1$, $\sin b > 1$, $\sin B > 1$, impossible.

6. Given $A = 124°$, and $a = 112°$.

Results,

$\sin h > 1$, $\sin b > 1$, $\sin B > 1$, impossible.

7. Given $A = 43°$, and $a = 43°$.

Results,

$h = 90°$, $b = 90°$, $B = 90°$.

8. Given $A = 43°$, and $a = 137°$.

Results,

$h = 90°$, $b = 270°$, $B = 270°$.

III.

265. Given A and b, to find a, h, and B.

By [173] or Napier's rules,

For solution,

$$\tan a = \tan A \sin b \quad \text{(i)}$$
$$\cot h = \cos A \cot b \quad \text{(ii)}$$
$$\cos B = \sin A \cos b \quad \text{(iii)}$$

$$a_2 = 180° + a_1$$
$$h_2 = 180° + h_1$$
$$B_2 = 360° - B_1$$

For a test,

$$\tan a \cot h = \cos B \quad . \; . \; . \; . \; . \; . \; . \; . \qquad \text{(iv)}$$

For discriminating between values,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \sin h \quad . \; . \; . \; . \; . \; . \; . \; . \qquad \text{(v)}$$

266. Each equation for solution gives two values of a required part, one less and one greater than 180°. The equations (v) chosen for discriminating between these values show that the three values which are less than 180° belong together, since they make each member of those equations positive; and the three values which exceed 180° belong together since they make each member negative.

267. The geometrical construction is made by drawing a great-circle arc $\mathbf{A\,C}$ equal to the given value of b; constructing at $\mathbf{C}$ a right angle, and at $\mathbf{A}$ an angle equal to the given value of A; then extending the sides of these angles until they cross twice, in points $\mathbf{B}_1$ and $\mathbf{B}_2$ diametrically opposite each other. Thus are formed two spherical right triangles $\mathbf{A\,B}_1\,\mathbf{C}$ and $\mathbf{A\,B}_2\,\mathbf{C}$, each containing with the given parts a different value of each required part. The two values of a are $\mathbf{C\,B}_1$ and $\mathbf{C\,B}_2$, the latter exceeding the former by a semi-circumference. The two values of h are $\mathbf{A\,B}_1$ and $\mathbf{A\,B}_2$, the latter exceeding the former by a semi-circumference. The two values of B are one at $\mathbf{B}_1$ less than 180° and one at $\mathbf{B}_2$ greater than 180°.

268. Examples.

1. Given $A = 37^\circ\ 40'$, and $b = 42^\circ\ 20'$.

Results,

$$\left.\begin{aligned} a &= 27^\circ\ 28'\ \ 6'' \\ h &= 49^\circ\ \ 0'\ 46'' \\ B &= 63^\circ\ \ 8'\ 43'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 207^\circ\ 28'\ \ 6'' \\ h &= 229^\circ\ \ 0'\ 46'' \\ B &= 296^\circ\ 51'\ 17'' \end{aligned}\right.$$

2. Given $A = 135° \, 27'$, and $b = 74° \, 18'$.

Results,

$$\left.\begin{aligned} a &= 136° \, 32' \, 17'' \\ h &= 101° \, 19' \, 38'' \\ B &= 79° \, 3' \, 24'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 316° \, 32' \, 17'' \\ h &= 281° \, 19' \, 38'' \\ B &= 280° \, 56' \, 36'' \end{aligned}\right.$$

3. Given $A = 67° \, 25'$, and $b = 138° \, 52'$.

Results,

$$\left.\begin{aligned} a &= 57° \, 41' \, 45'' \\ h &= 113° \, 44' \, 7'' \\ B &= 134° \, 3' \, 42'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 237° \, 41' \, 45'' \\ h &= 293° \, 44' \, 7'' \\ B &= 225° \, 56' \, 18'' \end{aligned}\right.$$

4. Given $A = 124° \, 20'$, and $b = 112° \, 40'$.

Results,

$$\left.\begin{aligned} a &= 126° \, 30' \, 28'' \\ h &= 76° \, 44' \, 45'' \\ B &= 108° \, 33' \, 21'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 306° \, 30' \, 28'' \\ h &= 256° \, 44' \, 45'' \\ B &= 251° \, 26' \, 39'' \end{aligned}\right.$$

IV.

269. Given h and a, to find A, b, and B.

By [173] or Napier's rules,

For solution,

$$\sin A = \csc h \sin a \quad \ldots\ldots \quad \text{(i)}$$

$$\cos b = \cos h \sec a \quad \ldots\ldots \quad \text{(ii)}$$

$$\cos B = \cot h \tan a \quad \ldots\ldots \quad \text{(iii)}$$

$$\begin{aligned} A_2 &= 180° - A_1 \\ b_2 &= 360° - b_1 \\ B_2 &= 360° - B_1 \end{aligned}$$

For a test,

$$\sin A \cos b = \cos B \quad \ldots\ldots \quad \text{(iv)}$$

For discriminating between values,

$$\sin b = \frac{\tan a}{\tan A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{v})$$

$$\sin B = \frac{\cos A}{\cos a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{vi})$$

270. The first equation gives supplementary values of A, each less than 180°. The second and third give one value less, and one value greater than 180° of b and of B. From the discriminating equations it appears that the values of b and B which are less than 180° are to be taken with that value of A which is in the same quadrant (first or second) with the given a.

271. The geometrical construction is as follows: Draw two great-circle arcs forming a right angle; on one side of which mark off $\mathbf{CB}$ equal to the given value of a, and extend the other side b to form a complete circumference. This circumference will, in general, be cut in two points $\mathbf{A}_1$ and $\mathbf{A}_2$ by the circumference of a small circle drawn from the point $\mathbf{B}$ as a pole and with a polar radius equal to the given value of h. Thus are formed two spherical right triangles $\mathbf{A}_1\mathbf{BC}$ and $\mathbf{A}_2\mathbf{BC}$, each containing with the given parts a different value of each required part. The two values of A are found at $\mathbf{A}_1$ and at $\mathbf{A}_2$, the angle at the latter point being the supplement of that at the former. The two values of B are the angle less than 180° between the arcs $\mathbf{BC}$ and $\mathbf{BA}_1$ and the angle greater than 180° between the arcs $\mathbf{BC}$ and $\mathbf{BA}_2$. The two values of b are the arc $\mathbf{CA}_1$ less than a semi-circumference and the arc $\mathbf{CA}_2$ greater than a semi-circumference.

272. From this construction it is evident that the problem is impossible when, a being less than 90°, h is less than a or greater than $180° - a$; and also when, a being greater than 90°, h is greater than a or less than $180° - a$. Illustrations occur among the examples.

273. Examples.

1. Given $h = 68°\ 37'$, and $a = 43°\ 25'$.

Results,

$$\left.\begin{aligned} A &= 47°\ 34'\ 15'' \\ b &= 59°\ 52'\ 16'' \\ B &= 68°\ 15'\ 15'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} A &= 132°\ 25'\ 45'' \\ b &= 300°\ \ 7'\ 44'' \\ B &= 291°\ 44'\ 45'' \end{aligned}\right.$$

2. Given $h = 108°\ 45'$, and $a = 49°\ 20'$.

Results,

$$\left.\begin{aligned} A &= 53°\ 13'\ 36'' \\ b &= 119°\ 33'\ 19'' \\ B &= 113°\ 16'\ 23'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} A &= 126°\ 46'\ 24'' \\ b &= 240°\ 26'\ 41'' \\ B &= 246°\ 43'\ 37'' \end{aligned}\right.$$

3. Given $h = 40°$, and $a = 150°$.

Results,

$$\left.\begin{aligned} A &= 128°\ 56'\ \ 6'' \\ b &= 152°\ 11'\ 43'' \\ B &= 133°\ 28'\ 37'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} A &= 51°\ \ 3'\ 54'' \\ b &= 207°\ 48'\ 17'' \\ B &= 226°\ 31'\ 23'' \end{aligned}\right.$$

4. Given $h = 100°$, and $a = 150°$.

Results,

$$\left.\begin{aligned} A &= 149°\ 29'\ 17'' \\ b &= 78°\ 25'\ 59'' \\ B &= 84°\ \ 9'\ 25'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} A &= 30°\ 30'\ 43'' \\ b &= 281°\ 34'\ \ 1'' \\ B &= 275°\ 50'\ 35'' \end{aligned}\right.$$

5. Given $h = 50°$, and $a = 70°$;
or $h = 120°$, and $a = 70°$;
or $h = 150°$, and $a = 130°$;
or $h = 40°$, and $a = 130°$.

Results, $\sin A > 1$; $\cos b$ either greater than 1 or less than -1; $\cos B$ either greater than 1 or less than -1. All these triangles are impossible.

6. Given $h = 65°$, and $a = 115°$. (Supplementary values.)

Results,

$$\begin{array}{lll} \sin A = 1 & \cos b = -1 & \cos B = -1 \\ A = 90° & b = 180° & B = 180° \end{array}$$

The small circle of polar radius h and pole **B** *touches* the circumference of the great circle b at a point diametrically opposite to **C**; here the two points **A₁** and **A₂** fall together, and the two triangles become one, and this is a lunary surface covering a half of the hemisphere.

7. Given $h = 70°$, and $a = 70°$. (Equal values.)

Results,

$$\begin{array}{lll} \sin A = 1 & \cos b = 1 & \cos B = 1 \\ A = 90° & b = 0° \text{ or } 360° & B = 0° \text{ or } 360° \end{array}$$

The small circle touches at **C**. One triangle is reduced to the arc **C B**; the other covers the whole hemisphere.

V.

274. Given a and b, to find h, A, and B.

By [173] or Napier's rules,

For solution,

$$\cos h = \cos a \cos b \quad \ldots\ldots \quad \text{(i)}$$

$$\cot A = \cot a \sin b \quad \ldots\ldots \quad \text{(ii)}$$

$$\cot B = \sin a \cot b \quad \ldots\ldots \quad \text{(iii)}$$

$$h_2 = 360° - h_1$$

$$A_2 = 180° + A_1$$

$$B_2 = 180° + B_1$$

For a test,

$$\cot A \cot B = \cos h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iv)}$$

For discriminating between values,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \sin h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(v)}$$

275. Each equation for solution gives two values of a required part, one less and one greater than 180°. The equations chosen for discriminating between these values show that the three values which are less than 180° belong together, since they make each member of those equations positive. The other three values make each member negative.

276. The geometrical meaning of these results is that when the two arcs *a* and *b* have been drawn forming a right angle at **C**, their other ends, **B** and **A** are joined either by an arc less than a semi-circumference or by an arc greater than a semi-circumference, the two arcs together making the whole circumference through **A** and **B**. In the first triangle the angles at **A** and **B** are each less than 180°; and in the second triangle they are each greater than 180°, being equal to those of the first triangle with 180° added to each.

277. Examples.

1. Given $a = 62° \; 14'$, and $b = 74° \; 26'$.

Results,

$$\left.\begin{aligned} h &= 82° \; 49' \; \; 5'' \\ A &= 63° \; \; 6' \; 23'' \\ B &= 76° \; \; 9' \; \; 9'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} h &= 277° \; 10' \; 55'' \\ A &= 243° \; \; 6' \; 23'' \\ B &= 256° \; \; 9' \; \; 9'' \end{aligned}\right.$$

2. Given $a = 58^\circ\ 32'$, and $b = 127^\circ\ 43'$.

Results,

$$\left.\begin{aligned} h &= 108^\circ\ 37'\ 22'' \\ A &= 64^\circ\ 10'\ 4'' \\ B &= 123^\circ\ 24'\ 36'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} h &= 251^\circ\ 22'\ 38'' \\ A &= 244^\circ\ 10'\ 4'' \\ B &= 303^\circ\ 24'\ 36'' \end{aligned}\right.$$

3. Given $a = 130^\circ$, and $b = 130^\circ$.

Results,

$$\left.\begin{aligned} h &= 65^\circ\ 35'\ 43'' \\ A &= 122^\circ\ 43'\ 56'' \\ B &= 122^\circ\ 43'\ 56'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} h &= 294^\circ\ 24'\ 17'' \\ A &= 302^\circ\ 43'\ 56'' \\ B &= 302^\circ\ 43'\ 56'' \end{aligned}\right.$$

VI.

278. Given A and B, to find h, a, and b.

By [173] or Napier's rules,

For solution,

$$\cos h = \cot A \cot B \quad . \ . \ . \ . \ . \ . \ . \quad \text{(i)}$$

$$\cos a = \cos A \csc B \quad . \ . \ . \ . \ . \ . \ . \quad \text{(ii)}$$

$$\cos b = \csc A \cos B \quad . \ . \ . \ . \ . \ . \ . \quad \text{(iii)}$$

$$\begin{aligned} h_2 &= 360^\circ - h_1 \\ a_2 &= 360^\circ - a_1 \\ b_2 &= 360^\circ - b_1 \end{aligned}$$

For a test,

$$\cos a \cos b = \cos h \quad . \ . \ . \ . \ . \ . \ . \ . \quad \text{(iv)}$$

For discriminating between values,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \sin h \quad . \ . \ . \ . \ . \ . \ . \quad \text{(v)}$$

279. Each equation for solution gives two values of a required part, one less and one greater than 180°. The discriminating equations show that the three values which are less than 180° belong together.

280. The geometrical meaning of these results is this: When the arcs h, a, and b, each being less than a semicircumference form a triangle containing the given angles A, B, and C, the arcs $360^\circ - h$, $360^\circ - a$, and $360^\circ - b$, join the vertices of this same triangle forming angles which are vertically opposite A, B, and C, respectively, and therefore equal to them.

281. Examples.

1. Given $A = 52^\circ\ 17'$, and $B = 121^\circ\ 28'$.

Results,

$$\left.\begin{aligned} h &= 118^\circ\ 14'\ 55'' \\ a &= 44^\circ\ 10'\ 25'' \\ b &= 131^\circ\ 17'\ 28'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} h &= 241^\circ\ 45'\ 5'' \\ a &= 315^\circ\ 49'\ 35'' \\ b &= 228^\circ\ 42'\ 32'' \end{aligned}\right.$$

2. Given $A = 117^\circ\ 46'$, and $B = 105^\circ\ 34'$.

Results,

$$\left.\begin{aligned} h &= 81^\circ\ 33'\ 57'' \\ a &= 118^\circ\ 55'\ 18'' \\ b &= 107^\circ\ 39'\ 18'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} h &= 278^\circ\ 26'\ 3'' \\ a &= 241^\circ\ 4'\ 42'' \\ b &= 252^\circ\ 20'\ 42'' \end{aligned}\right.$$

3. Given $A = 90^\circ$, and $B = 50^\circ$.

Results,

$$\left.\begin{aligned} h &= 90^\circ \\ a &= 90^\circ \\ b &= 50^\circ \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} h &= 270^\circ \\ a &= 270^\circ \\ b &= 310^\circ \end{aligned}\right.$$

CHAPTER X.

THE SOLUTION OF SPHERICAL OBLIQUE TRIANGLES.

282. The given parts of a spherical oblique triangle may be:

I. Two sides and the included angle.
II. Two angles and the included side.
III. Two sides and the angle opposite one of them.
IV. Two angles and the side opposite one of them.
V. The three sides.
VI. The three angles.

283. The problem always has two solutions; that is, there are always two triangles each of which contains with the given parts a different value of each required part. In the examples that follow, the given parts are limited not to exceed 180°, but results exceeding that limit are retained and interpreted.

I.

284. Given two sides and the included angle, a, b, and C.

The geometric construction on a globe is as follows: Draw two great-circle arcs, equal to the given a and b respectively, forming with each other an angle equal to the given C; and join the ends **B** and **A** of these arcs either by an arc c_1 less than 180° or by an arc c_2 greater than 180°. The angles A_1 and B_1 formed by c_1 with b and a, are each less than 180°; but the angles A_2 and B_2, formed by c_2 with b and a, are each greater than 180°, being equal respectively to $A_1 + 180°$ and $B_1 + 180°$. This construction is always possible; for, when arcs **CB** and **CA** of any lengths whatever, and forming at **C** any angle whatever have been drawn, it only remains to draw the circumference of a great circle through the two points B and A, which can always be done.

285. First Method. By the fundamental equations [158] slightly changed in form, and [152],

$$\cot A = \frac{\cos a \sin b - \sin a \cos b \cos C}{\sin a \sin C}$$

$$\cot B = \frac{\sin a \cos b - \cos a \sin b \cos C}{\sin b \sin C}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

$$A_2 = 180^\circ + A_1$$

$$B_2 = 180^\circ + B_1$$

$$c_2 = 360^\circ - c_1$$

Which of these values belong together as parts of one triangle and which as parts of the other is decided by means of the equations

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

The computed values of A, B, and c, which, with the given values of a, b, and C, make each of these ratios positive, belong to one triangle, and those which make each ratio negative belong to the other triangle. In other words, one triangle has a positive, the other a negative modulus.

286. Examples.

1. Given, one side = 119°, another = 116°, and the included angle = 133°.

By Table I. (p. 9),

$a = 119^\circ$	$\sin a = 0.875$	$\cos a = -0.485$
$b = 116^\circ$	$\sin b = 0.899$	$\cos b = -0.438$
$C = 133^\circ$	$\sin C = 0.731$	$\cos C = -0.682$

By substitution in the equations,

$$\cot A = \frac{(-0.485)(0.899)-(0.875)(-0.438)(-0.682)}{(0.875)(0.731)} = -1.090$$

$$\cot B = \frac{(0.875)(-0.438)-(-0.485)(0.899)(-0.682)}{(0.899)(0.731)} = -1.036$$

$$\cos c = (-0.485)(-0.438)+(0.875)(0.899)(-0.682) = -0.324$$

Whence

$$\left.\begin{aligned} A &= 137^\circ\ 28' \\ B &= 136^\circ\ \ 0' \\ c &= 108^\circ\ 53' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} A &= 317^\circ\ 28' \\ B &= 316^\circ\ \ 0' \\ c &= 251^\circ\ \ 7' \end{aligned}\right.$$

For results obtained by five-place logarithms, see p. 208, and p. 213.

2. Given, one side = 112°, another = 80°, and the included angle = 76°.

Results,

$$\left.\begin{aligned} A &= 114^\circ\ 26' \\ B &= \ \ 75^\circ\ 19' \\ c &= \ \ 81^\circ\ \ 0' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} A &= 294^\circ\ 26' \\ B &= 255^\circ\ 19' \\ c &= 279^\circ\ \ 0' \end{aligned}\right.$$

For results obtained by five-place logarithms, see p. 209.

287. SECOND METHOD. A convenient method for logarithmic computation is based on the Gaussian Equations [171], which, by taking the logarithms of each term, may be written thus:

$$\text{(i)}\quad \log\sin\tfrac{1}{2}(A+B)+\log\cos\tfrac{1}{2}c=\log\cos\tfrac{1}{2}(a-b)+\log\cos\tfrac{1}{2}C$$
$$\text{(ii)}\quad \log\sin\tfrac{1}{2}(A-B)+\log\sin\tfrac{1}{2}c=\log\sin\tfrac{1}{2}(a-b)+\log\cos\tfrac{1}{2}C$$
$$\text{(iii)}\quad \log\cos\tfrac{1}{2}(A+B)+\log\cos\tfrac{1}{2}c=\log\cos\tfrac{1}{2}(a+b)+\log\sin\tfrac{1}{2}C$$
$$\text{(iv)}\quad \log\cos\tfrac{1}{2}(A-B)+\log\sin\tfrac{1}{2}c=\log\sin\tfrac{1}{2}(a+b)+\log\sin\tfrac{1}{2}C$$

In the second members are found only given quantities, and in the first members only required quantities. How to find the values of the latter by combining these equations will be shown by examples.

288. Examples.

1. Given, one side $= 119°$, another $= 116°$, and the included angle $= 133°$.

$a = 119°$
$b = 116°$
$C = 133°$

$\frac{1}{2}(a+b) = 117° 30'$	log sin	9.94793	(1)
	log cos	9.66441*n*	(2)
$\frac{1}{2}(a-b) = 1° 30'$	log sin	8.41792	(3)
	log cos	9.99985	(4)
$\frac{1}{2} C = 66° 30'$	log sin	9.96240	(5)
	log cos	9.60070	(6)

(i)	$\log\sin\frac{1}{2}(A+B) + \log\cos\frac{1}{2}c =$	9.60055	(7) = (4) + (6)
(ii)	$\log\sin\frac{1}{2}(A-B) + \log\sin\frac{1}{2}c =$	8.01862	(8) = (3) + (6)
(iii)	$\log\cos\frac{1}{2}(A+B) + \log\cos\frac{1}{2}c =$	9.62681*n*	(9) = (2) + (5)
(iv)	$\log\cos\frac{1}{2}(A-B) + \log\sin\frac{1}{2}c =$	9.91033	(10) = (1) + (5)

$\log\tan\frac{1}{2}(A+B)$	=	9.97374*n*	(11) = (7) — (9)
$\log\tan\frac{1}{2}(A-B)$	=	8.10829	(12) = (8) — (10)
$\log\cos\frac{1}{2}(A+B)$	=	9.86222*n*	(13)
$\log\cos\frac{1}{2}(A-B)$	=	9.99996	(14)
$\log\cos\frac{1}{2}c$	=	9.76459	(15) = (9) — (13)
$\log\sin\frac{1}{2}c$	=	9.91037	(16) = (10) — (14)
$\log\tan\frac{1}{2}c$	=	10.14578	(17) = (16) — (15)

$\frac{1}{2}(A+B) = 136° 43' 53''$
$\frac{1}{2}(A-B) = 0° 44' 7''$
$\frac{1}{2}c = 54° 26' 27''$

$A_1 = 137° 28' 0''$	$A_2 = 317° 28' 0''$
$B_1 = 135° 59' 46''$	$B_2 = 315° 59' 46''$
$c_1 = 108° 52' 54''$	$c_2 = 251° 7' 6''$

Test:

log sin *a*, *b*, *c*,	9.94182	9.95366	9.97598
log sin *A*, *B*, *C*,	9.82996	9.84180	9.86413
log sin modulus,	0.11186	0.11186	0.11185

The course of the computation is indicated by the reference figures at the right. Numbers (1) to (6) are logarithms taken out of the table. Numbers (7) to (10) are the results of adding these logarithms two and two in the order indicated; and are, therefore, the second members of the equations (i), (ii), (iii), and (iv) respectively. Numbers (11) and (12) result from subtracting the members of (iii) from those of (i), and those of (iv) from those of (ii). With (11) and (12) we turn to the table and take out the values of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$, also, at the same places, numbers (13) and (14), or log cos $\frac{1}{2}(A+B)$ and log cos $\frac{1}{2}(A-B)$. These last subtracted from (9) and (10) respectively, give numbers (15) and (16) from which or from whose difference, number (17), the arc $\frac{1}{2}c$ is found.

The results of this computation give one set of values for A, B, and c, namely, those which do not exceed 180°. The values of A_2, B_2, and c_2, the parts which exceed 180°, are easily found from their geometric relations to A_1, B_1, c_1. For these relations, see Arts. 284 and 285.

To test the computation, we take out the *log sines* of a, b, and c, also of A, B, and C; and subtract the last three from the first three respectively, thus finding the *log of the modulus* three times, independently.

2. Given, one side = 112°, another = 80°, and the included angle = 76°.

Results,

$$\left.\begin{aligned} A &= 114^\circ\ 23'\ 17'' \\ B &= 75^\circ\ 19'\ 27'' \\ c &= 81^\circ\ 2'\ 4'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} A &= 294^\circ\ 23'\ 17'' \\ B &= 255^\circ\ 19'\ 27'' \\ c &= 278^\circ\ 57'\ 56'' \end{aligned}\right.$$

289. Third Method. Another method for logarithmic computation is based on Napier's Analogies [172], which, by taking the logarithm of each term, may be written thus:

(iv) $\log\tan\frac{1}{2}(A+B)=\log\sec\frac{1}{2}(a+b)+\log\cos\frac{1}{2}(a-b)+\log\cot\frac{1}{2}C$

(iii) $\log\tan\frac{1}{2}(A-B)=\log\csc\frac{1}{2}(a+b)+\log\sin\frac{1}{2}(a-b)+\log\cot\frac{1}{2}C$

(ii) $\log\tan\frac{1}{2}c=\log\cos\frac{1}{2}(A+B)+\log\sec\frac{1}{2}(A-B)+\log\tan\frac{1}{2}(a+b)$

(i) $\log\tan\frac{1}{2}c=\log\sin\frac{1}{2}(A+B)+\log\csc\frac{1}{2}(A-B)+\log\tan\frac{1}{2}(a-b)$

The second members of (iv) and (iii) contain only known quantities, so that their first members give $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$, and thence A and B. When the value of $\frac{1}{2}(A+B)$ is taken out of the table, its *log cos* and *log sin* are taken out also; and when that of $\frac{1}{2}(A-B)$ is taken out its *log sec* and *log csc* are taken out also. These functions so taken out are substituted in the second members of (ii) and (i), so that $\frac{1}{2}c$ may be computed in two ways; and the results should agree.

290. Examples.

Given, one side = 123° 15′, another = 108° 24′, and the included angle = 98° 32′.

$a=123°\,15'$

$b=108°\,24'$

$C=98°\,32'$	(iv)	(iii)
$\frac{1}{2}(a+b)=115°\,49'\,30''$	log sec 0.36089n	log csc 0.04570
$\frac{1}{2}(a-b)=7°\,25'\,30''$	log cos 9.99634	log sin 9.11135
$\frac{1}{2}C=49°\,16'$	log cot 9.93508	log cot 9.93508
$\frac{1}{2}(A+B)=117°\,1'\,41''$	log tan 10.29231n	
$\frac{1}{2}(A-B)=7°\,2'\,52''$		log tan 9.09213

	(ii)	(i)
$\frac{1}{2}(A+B)$	log cos 9.65746n	log sin 9.94977
$\frac{1}{2}(A-B)$	log sec 0.00330	log csc 0.91117
$\frac{1}{2}(a+b)$	log tan 10.31519n	
$\frac{1}{2}(a-b)$		log tan 9.11501
$\frac{1}{2}c=43°\,24'\,53''$	log tan 9.97595	log tan 9.97595

$A_1 = 124°\ \ 4'\ 33''$ $\qquad A_2 = 304°\ \ 4'\ 33''$

$B_1 = 109°\ 58'\ 49''$ $\qquad B_2 = 289°\ 58'\ 49''$

$c_1 = \ 86°\ 49'\ 46''$ $\qquad c_2 = 273°\ 10'\ 14''$

Test:

log sin *a*, *b*, *c*,	9.92235	9.97721	9.99934
log sin *A*, *B*, *C*,	9.91819	9.97304	9.99517
log modulus,	0.00416	0.00417	0.00417

291. Fourth Method. By drawing from **B** or from **A** a great-circle arc perpendicular to the opposite side, the required triangle is made equivalent to the sum or to the difference of two spherical right triangles, which can be solved as follows:

The perpendicular **B Q** forms two right triangles **B Q C** and **B Q A**, the former of which has the hypothenuse *a* and an oblique angle *C* given. Then, to find the base **C Q** (Art. 256),

$$\tan \mathrm{CQ} = \tan a \cos C \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The base **Q A** of the other right triangle is the given side *b* diminished by **C Q**.

$$\mathrm{QA} = b - \mathrm{CQ} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

But if **C Q** is numerically greater than *b* the base

$$\mathrm{QA} = 360° + b - \mathrm{CQ}$$

It is then the longer arc from **Q** to **A**, as may be seen by drawing the triangle on a globe. Napier's first rule (Art. 248), applied to the two right triangles **B Q C** and **B Q A**, gives the two equations

$$\sin \mathrm{CQ} = \cot C \tan \mathrm{BQ}$$
$$\sin \mathrm{QA} = \cot A \tan \mathrm{BQ}$$

whence, by eliminating tan **B Q**,

$$\cot A = \cot C \csc \mathrm{CQ} \sin \mathrm{QA} \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Napier's second rule, applied to the same two triangles, gives the two equations

$$\cos a = \cos \mathrm{CQ} \cos \mathrm{BQ}$$
$$\cos c = \cos \mathrm{QA} \cos \mathrm{BQ}$$

whence, by eliminating $\cos \mathrm{BQ}$,

$$\cos c = \cos a \sec \mathrm{CQ} \cos \mathrm{QA} \quad . \ . \ . \ . \ . \ . \quad \text{(iv)}$$

The perpendicular from **A** forms two right triangles **APC** and **APB**, from which in the same way come the equations

$$\tan \mathrm{CP} = \tan b \cos C \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{(v)}$$

$$\left.\begin{array}{rl} \mathrm{PB} & = a - \mathrm{CP} \\ \text{or } \mathrm{PB} & = 360^\circ + a - \mathrm{CP} \end{array}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \quad \text{(vi)}$$

$$\cot B = \cot C \csc \mathrm{CP} \sin \mathrm{PB} \quad . \ . \ . \ . \quad \text{(vii)}$$

$$\cos c = \cos b \sec \mathrm{CP} \cos \mathrm{PB} \quad . \ . \ . \ . \quad \text{(viii)}$$

292. Examples.

1. Given, one side = 119°, another side = 116°, and the included angle = 133°.

$$\begin{aligned} a &= 119^\circ \\ b &= 116^\circ \\ C &= 133^\circ \end{aligned}$$

(i)
$$\begin{aligned} \log \tan a &= 10.25625n \\ \log \cos C &= \underline{\ 9.83378n} \\ \log \tan \mathrm{CQ} &= 10.09003 \end{aligned}$$

$$\mathrm{CQ} = 50^\circ\ 53'\ 49''$$

(ii) $\mathrm{QA} = 65^\circ\ 6'\ 11''$

(iii)
$$\begin{aligned} \log \cot C &= 9.96966n \\ \log \csc \mathrm{CQ} &= 0.11013 \\ \log \sin \mathrm{QA} &= \underline{9.95764} \\ \log \cot A &= 10.03743n \end{aligned}$$

(v)
$$\begin{aligned} \log \tan b &= 10.31182n \\ \log \cos C &= \underline{\ 9.83378n} \\ \log \tan \mathrm{CP} &= 10.14560 \end{aligned}$$

$$\mathrm{CP} = 54^\circ\ 25'\ 46''$$

(vi) $\mathrm{PB} = 64^\circ\ 34'\ 14''$

(vii)
$$\begin{aligned} \log \cot C &= 9.96966n \\ \log \csc \mathrm{CP} &= 0.08969 \\ \log \sin \mathrm{PB} &= \underline{9.95574} \\ \log \cot B &= 10.01509n \end{aligned}$$

$$\begin{aligned} A_1 &= 137^\circ\ 27'\ 58'' \\ A_2 &= 317^\circ\ 27'\ 58'' \end{aligned}$$

$$\begin{aligned} B_1 &= 135^\circ\ 59'\ 43'' \\ B_2 &= 315^\circ\ 59'\ 43'' \end{aligned}$$

(iv)
$$\begin{aligned} \log\cos a &= 9.68557n \\ \log\sec \mathrm{CQ} &= 0.20016 \\ \log\cos \mathrm{QA} &= \underline{9.62427} \\ \log\cos c &= 9.51000n \end{aligned}$$

$$\begin{aligned} c_1 &= 108^\circ\ 52'\ 49'' \\ c_2 &= 251^\circ\ \ 7'\ 11'' \end{aligned}$$

(viii)
$$\begin{aligned} \log\cos b &= 9.64184n \\ \log\sec \mathrm{CP} &= 0.23530 \\ \log\cos \mathrm{PB} &= \underline{9.63286} \\ \log\cos c &= 9.51000n \end{aligned}$$

Test:

log sin a, b, c,	9.94182	9.95366	9.97598
log sin A, B, C,	9.82996	9.84181	9.86413
log modulus,	0.11186	0.11185	0.11185

2. Given, one side = 62° 15′, another side = 34° 28′, and the included angle = 156° 34′.

$$\begin{aligned} a &= 62^\circ\ 15' \\ b &= 34^\circ\ 28' \\ C &= 156^\circ\ 34' \end{aligned}$$

(i)
$$\begin{aligned} \log\tan a &= 10.27891 \\ \log\cos C &= \underline{\ 9.96262n} \\ \log\tan \mathrm{CQ} &= 10.24153n \end{aligned}$$

$$\mathrm{CQ} = 119^\circ\ 49'\ 50''$$

(ii) $\mathrm{QA} = 274^\circ\ 38'\ 10''$

(iii)
$$\begin{aligned} \log\cot C &= 10.36308n \\ \log\csc \mathrm{CQ} &= 0.06173 \\ \log\sin \mathrm{QA} &= \underline{\ 9.99858n} \\ \log\cot A &= 10.42339 \end{aligned}$$

$$\begin{aligned} A_1 &= 20^\circ\ 40'\ 5'' \\ A_2 &= 200^\circ\ 40'\ 5'' \end{aligned}$$

(iv)
$$\begin{aligned} \log\cos a &= 9.66803 \\ \log\sec \mathrm{CQ} &= 0.30327n \\ \log\cos \mathrm{QA} &= \underline{8.90756} \\ \log\cos c &= 8.87886n \end{aligned}$$

(v)
$$\begin{aligned} \log\tan b &= 9.83659 \\ \log\cos C &= \underline{9.96262n} \\ \log\tan \mathrm{CP} &= 9.79921n \end{aligned}$$

$$\mathrm{CP} = 147^\circ\ 47'\ 49''$$

(vii) $\mathrm{PB} = 274^\circ\ 27'\ 11''$

(vii)
$$\begin{aligned} \log\cot C &= 10.36308n \\ \log\csc \mathrm{CP} &= 0.27333 \\ \log\sin \mathrm{PB} &= \underline{\ 9.99869n} \\ \log\cot B &= 10.63510 \end{aligned}$$

$$\begin{aligned} B_1 &= 13^\circ\ 2'\ 40'' \\ B_2 &= 193^\circ\ 2'\ 40'' \end{aligned}$$

(viii)
$$\begin{aligned} \log\cos b &= 9.91617 \\ \log\sec \mathrm{CP} &= 0.07254n \\ \log\cos \mathrm{PB} &= \underline{8.89010} \\ \log\cos c &= 8.87881n \end{aligned}$$

$c_1 = \ \ 94° \ 20' \ 21''$
$c_2 = 265° \ 39' \ 39''$

Test:

log sin a, b, c,	9.94694	9.75276	9.99876
log sin A, B, C,	9.54772	9.35355	9.59954
log modulus,	0.39922	0.39921	0.39922

3. Given, one side = 123°, another side = 82°, and the included angle = 65°.

Results,

$$\left.\begin{aligned} A &= 127° \ 45' \ 23'' \\ B &= \ \ 68° \ 59' \ 31'' \\ c &= \ \ 74° \ \ 1' \ 36'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} A &= 307° \ 45' \ 23'' \\ B &= 248° \ 59' \ 31'' \\ c &= 285° \ 58' \ 24'' \end{aligned}\right.$$

II.

293. Given two angles and the included side, A, B, and c.

The geometric construction on a globe is as follows: Draw a great-circle arc equal to the given c, at one end of which construct an angle equal to the given A, and, on the same side of c, but at the other end, an angle equal to the given B; then extend the sides of these angles till they cross once in a point $\mathbf{C}_1$ and again in a point $\mathbf{C}_2$ diametrically opposite $\mathbf{C}_1$. The sides a_1 and b_1 meeting at $\mathbf{C}_1$ are each less than 180°; but the sides a_2 and b_2 meeting at $\mathbf{C}_2$ are each greater than 180°, being equal respectively to $a_1 + 180°$ and $b_1 + 180°$. The angle at $\mathbf{C}_1$ is less than 180°, but the angle at $\mathbf{C}_2$ is greater, being equal to $360° - C_1$. This construction is always possible; for two great-circle arcs can be drawn making any angles whatever with a third arc, and the two arcs, if extended far enough, cross each other in two points.

294. FIRST METHOD. By the fundamental equations [158] and [155],

$$\cot a = \frac{\cos A \sin B + \sin A \cos B \cos c}{\sin A \sin c}$$

$$\cot b = \frac{\sin A \cos B + \cos A \sin B \cos c}{\sin B \sin c}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

$$a_2 = 180° + a_1$$
$$b_2 = 180° + b_1$$
$$C_2 = 360° - C_1$$

Which values belong together is decided by means of the equations

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

The computed values of a, b, and C, which, with the given values of A, B, and c, make each of these ratios positive belong to one triangle, and those which make each ratio negative belong to the other triangle. One triangle has a positive, the other a negative modulus.

295. Examples.

1. Given, one angle = 78°, another angle = 63°, and the included side = 127°.

By Table I. (p. 9).

$A = 78°$	$\sin A = 0.978$	$\cos A = 0.208$
$B = 63°$	$\sin B = 0.891$	$\cos B = 0.454$
$c = 127°$	$\sin c = 0.799$	$\cos c = -0.602$

By substitution in the formulas,

$$\cot a = \frac{(0.208)(0.891) + (0.978)(0.454)(-0.602)}{(0.978)(0.799)} = -0.105$$

$$\cot b = \frac{(0.978)(0.454) + (0.208)(0.891)(-0.602)}{(0.891)(0.799)} = 0.467$$

$$\cos C = -(0.208)(0.454) + (0.978)(0.891)(-0.602) = -0.619$$

$$\left.\begin{aligned} a &= 96^\circ\ 0' \\ b &= 64^\circ\ 57' \\ C &= 128^\circ\ 14' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 276^\circ\ 0' \\ b &= 244^\circ\ 57' \\ C &= 231^\circ\ 46' \end{aligned}\right.$$

2. Given, one angle = 118°, another = 67°, and the included side = 74°.

Results,

$$\left.\begin{aligned} a &= 111^\circ\ 39' \\ b &= 75^\circ\ 41' \\ C &= 65^\circ\ 56' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 291^\circ\ 39' \\ b &= 255^\circ\ 41' \\ C &= 294^\circ\ 4' \end{aligned}\right.$$

296. SECOND METHOD. For logarithmic computation, the Gaussian Equations [171] may be written with only known quantities in the second members thus:

(i) $\log\cos\frac{1}{2}(a-b) + \log\cos\frac{1}{2}C = \log\sin\frac{1}{2}(A+B) + \log\cos\frac{1}{2}c$

(ii) $\log\sin\frac{1}{2}(a-b) + \log\cos\frac{1}{2}C = \log\sin\frac{1}{2}(A-B) + \log\sin\frac{1}{2}c$

(iii) $\log\cos\frac{1}{2}(a+b) + \log\sin\frac{1}{2}C = \log\cos\frac{1}{2}(A+B) + \log\cos\frac{1}{2}c$

(iv) $\log\sin\frac{1}{2}(a+b) + \log\sin\frac{1}{2}C = \log\cos\frac{1}{2}(A-B) + \log\sin\frac{1}{2}c$

How to combine these equations so as to find the values of the required parts will be shown by examples.

297. Examples.

Given, one angle = 78°, another angle = 63°, and the included side = 127°.

$A = 78^\circ$
$B = 63^\circ$
$c = 127^\circ$

$\frac{1}{2}(A+B) = 70^\circ\ 30'$	log sin	9.97435	(1)
	log cos	9.52350	(2)
$\frac{1}{2}(A-B) = 7^\circ\ 30'$	log sin	9.11570	(3)
	log cos	9.99627	(4)
$\frac{1}{2}c = 63^\circ\ 30'$	log sin	9.95179	(5)
	log cos	9.64953	(6)

(i)	$\log\cos\frac{1}{2}(a-b) + \log\cos\frac{1}{2}C =$	9.62388	(7) = (1) + (6)
(ii)	$\log\sin\frac{1}{2}(a-b) + \log\cos\frac{1}{2}C =$	9.06749	(8) = (3) + (5)
(iii)	$\log\cos\frac{1}{2}(a+b) + \log\sin\frac{1}{2}C =$	9.17303	(9) = (2) + (6)
(iv)	$\log\sin\frac{1}{2}(a+b) + \log\sin\frac{1}{2}C =$	9.94806	(10) = (4) + (5)

$\log\tan\frac{1}{2}(a-b)$	= 9.44361	(11) = (8) − (7)
$\log\tan\frac{1}{2}(a+b)$	= 10.77503	(12) = (10) − (9)
$\log\cos\frac{1}{2}(a-b)$	= 9.98387	(13)
$\log\cos\frac{1}{2}(a+b)$	= 9.21893	(14)
$\log\cos\frac{1}{2}C$	= 9.64001	(15) = (7) − (13)
$\log\sin\frac{1}{2}C$	= 9.95410	(16) = (9) − (14)
$\log\tan\frac{1}{2}C$	= 10.31409	(17) = (16) − (15)

$\frac{1}{2}(a-b) = 15^\circ\ 31'\ 16''$
$\frac{1}{2}(a+b) = 80^\circ\ 28'\ 15''$
$\frac{1}{2}C = 64^\circ\ 7'\ 4''$

$a_1 = 95^\circ\ 59'\ 31''$	$a_2 = 275^\circ\ 59'\ 31''$
$b_1 = 64^\circ\ 56'\ 59''$	$b_2 = 244^\circ\ 56'\ 59''$
$C_1 = 128^\circ\ 14'\ 8''$	$C_2 = 231^\circ\ 45'\ 52''$

Test:

log sin a, b, c,	= 9.99762	9.95710	9.90235
log sin A, B, C,	= 9.99040	9.94988	9.89513
log modulus,	= 0.00722	0.00722	0.00722

The course of the computation is indicated by the reference numbers at the right hand. Numbers (1) to (6) are logarithms taken out of the table. Numbers (7) to (10) inclusive are the results of combining these logarithms two and two in the order indicated, and are therefore the second members of the equations (i), (ii), (iii), and (iv) respectively. Numbers (11) and (12) result from subtracting the members of (i) from those of (ii), and those of (iii) from those of (iv). With (11) and (12) we turn to the table and take out the values of $\frac{1}{2}(a-b)$ and $\frac{1}{2}(a+b)$; also, at the same places, numbers (13) and (14), or $\log\cos\frac{1}{2}(a-b)$ and $\log\cos\frac{1}{2}(a+b)$. These subtracted from (7) and (9) respectively give numbers (15) and (16) from which or from whose difference, number (17), the angle $\frac{1}{2}C$ is found.

The second set of values is derived from the first set by means of the geometric relations stated in Art. 293.

2. Given, one angle = 118°, another angle = 67°, and the included side = 74°.

Results,

$$\left.\begin{aligned} a &= 111^\circ\ 39'\ 31'' \\ b &= 75^\circ\ 40'\ 49'' \\ C &= 65^\circ\ 57'\ 22'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 291^\circ\ 39'\ 31'' \\ b &= 255^\circ\ 40'\ 49'' \\ C &= 294^\circ\ \ 2'\ 38'' \end{aligned}\right.$$

298. THIRD METHOD. Another method for logarithmic computation is based on Napier's Analogies [172], which, by taking the logarithm of each term, may be written thus:

$$\text{(i)}\quad \log\tan\tfrac{1}{2}(a-b) = \log\csc\tfrac{1}{2}(A+B) + \log\sin\tfrac{1}{2}(A-B) + \log\tan\tfrac{1}{2}c$$
$$\text{(ii)}\quad \log\tan\tfrac{1}{2}(a+b) = \log\sec\tfrac{1}{2}(A+B) + \log\cos\tfrac{1}{2}(A-B) + \log\tan\tfrac{1}{2}c$$
$$\text{(iii)}\quad \log\cot\tfrac{1}{2}C = \log\csc\tfrac{1}{2}(a-b) + \log\sin\tfrac{1}{2}(a+b) + \log\tan\tfrac{1}{2}(A-B)$$
$$\text{(iv)}\quad \log\cot\tfrac{1}{2}C = \log\sec\tfrac{1}{2}(a-b) + \log\cos\tfrac{1}{2}(a+b) + \log\tan\tfrac{1}{2}(A+B)$$

The two first equations give $\frac{1}{2}(a-b)$ and $\frac{1}{2}(a+b)$, and thence a and b. When the value of $\frac{1}{2}(a-b)$ is taken out of the table, its *log csc* and *log sec* are taken out also; and when that of $\frac{1}{2}(a+b)$ is taken out its *log sin* and *log cos* are taken out also. These substituted in (iii) and (iv) give $\frac{1}{2}C$ in two ways.

299. **Example.**

Given, one angle = 124° 12′, another = 82° 28′, and the included side = 64° 36′.

$A = 124° \, 12'$
$B = 82° \, 28'$
$c = 64° \, 36'$

		(i)		(ii)
$\frac{1}{2}(A+B) = 103° \, 20'$	log csc	0.01187	log sec	0.63711*n*
$\frac{1}{2}(A-B) = 20° \, 52'$	log sin	9.55169	log cos	9.97054
$\frac{1}{2}c = 32° \, 18'$	log tan	9.80084	log tan	9.80084
$\frac{1}{2}(a-b) = 13° \, 1' \, 48''$	log tan	9.36440		
$\frac{1}{2}(a+b) = 111° \, 19' \, 32''$			log tan	10.40849*n*

		(iii)		(iv)
$\frac{1}{2}(a-b)$	log csc	0.64693	log sec	0.01133
$\frac{1}{2}(a+b)$	log sin	9.96919	log cos	9.56070*n*
$\frac{1}{2}(A-B)$	log tan	9.58115		
$\frac{1}{2}(A+B)$			log tan	10.62524*n*
$\frac{1}{2}C = 32° \, 24' \, 47''$	log cot	10.19727	log cot	10.19727

$a_1 = 124° \, 21' \, 20''$	$a_2 = 304° \, 21' \, 20''$
$b_1 = 98° \, 17' \, 44''$	$b_2 = 278° \, 17' \, 44''$
$C_1 = 64° \, 49' \, 34''$	$C_2 = 295° \, 10' \, 26''$

Test:

log sin *a*, *b*, *c*,	9.91674	9.99543	9.95585
log sin *A*, *B*, *C*,	9.91755	9.99624	9.95666
log modulus,	9.99919	9.99919	9.99919

300. FOURTH METHOD. By drawing from B a great-circle arc perpendicular to the opposite side, the required triangle is made equivalent to the sum or to the difference of two right triangles, which may be solved as follows:

The perpendicular B Q forms two right triangles B Q A and B Q C, the former of which has the hypothenuse c and an oblique angle A given. Then to find the other oblique angle A B Q (Art. 256)

$$\cot \mathrm{ABQ} = \tan A \cos c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

The oblique angle Q B C in the other triangle is equal to the given angle B diminished by A B Q, or to $360° + B$ diminished by A B Q.

$$\left.\begin{array}{r} \mathrm{QBC} = B - \mathrm{ABQ} \\ \text{or } \mathrm{QBC} = 360° + B - \mathrm{ABQ} \end{array}\right\} \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

Napier's first rule (Art. 248), applied to the two right triangles B Q A and B Q C, gives the two equations

$$\cos \mathrm{ABQ} = \cot c \tan \mathrm{BQ}$$
$$\cos \mathrm{QBC} = \cot a \tan \mathrm{BQ}$$

whence, by eliminating tan B Q,

$$\cot a = \cot c \sec \mathrm{ABQ} \cos \mathrm{QBC} \quad . \quad . \quad . \qquad \text{(iii)}$$

Napier's second rule, applied to the same two triangles, gives the two equations,

$$\cos A = \sin \mathrm{ABQ} \cos \mathrm{BQ}$$
$$\cos C = \sin \mathrm{QBC} \cos \mathrm{BQ}$$

whence, by eliminating cos B Q,

$$\cos C = \cos A \csc \mathrm{ABQ} \sin \mathrm{QBC} \quad . \quad . \quad . \qquad \text{(iv)}$$

In the same way, from the two right triangles A P B and A P C, formed by drawing the perpendicular A P from A, come the equations

$$\cot \mathrm{BAP} = \tan B \cos c \quad . \; . \; . \; . \; . \quad \text{(v)}$$

$$\left.\begin{array}{r} \mathrm{PAC} = A - \mathrm{BAP} \\ \text{or } \mathrm{PAC} = 360^\circ + A - \mathrm{BAP} \end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad \text{(vi)}$$

$$\cot b = \cot c \sec \mathrm{BAP} \cos \mathrm{PAC} \quad . \; . \; . \; . \quad \text{(vii)}$$

$$\cos C = \cos B \csc \mathrm{BAP} \sin \mathrm{PAC} \quad . \; . \; . \quad \text{(viii)}$$

301. **Examples.**

1. Given, one angle = 118°, another = 67°, and the included side = 74°.

$A = 118^\circ$
$B = 67^\circ$
$c = 74^\circ$

(i) $\log \tan A = 10.27433n$
$\log \cos c = 9.44034$
$\log \cot \mathrm{ABQ} = 9.71467n$

$\mathrm{ABQ} = 117^\circ \; 24' \; 10''$

(ii) $\mathrm{QBC} = 309^\circ \; 35' \; 50''$

$\log \cot c = 9.45750$
$\log \sec \mathrm{ABQ} = 0.33701n$
(iii) $\log \cos \mathrm{QBC} = 9.80440$
$\log \cot a = 9.59891n$

$a_1 = 111^\circ \; 39' \; 31''$
$a_2 = 291^\circ \; 39' \; 31''$

(iv) $\log \cos A = 9.67161n$
$\log \csc \mathrm{ABQ} = 0.05169$
$\log \sin \mathrm{QBC} = 9.88680n$
$\log \cos C = 9.61010$

$C_1 = 65^\circ \; 57' \; 13''$ or $15''$
$C_2 = 294^\circ \; 2' \; 47''$ or $45''$

(v) $\log \tan B = 10.37215$
$\log \cos c = 9.44034$
$\log \cot \mathrm{BAP} = 9.81249$

$\mathrm{BAP} = 57^\circ \; 0' \; 6''$

(vi) $\mathrm{PAC} = 60^\circ \; 59' \; 54''$

(viii) $\log \cot c = 9.45750$
$\log \sec \mathrm{BAP} = 0.26391$
(vii) $\log \cos \mathrm{PAC} = 9.68559$
$\log \cot b = 9.40700$

$b_1 = 75^\circ \; 40' \; 48''$
$b_2 = 255^\circ \; 40' \; 48''$

(viii) $\log \cos B = 9.59188$
$\log \csc \mathrm{BAP} = 0.07640$
$\log \sin \mathrm{PAC} = 9.94181$
$\log \cos C = 9.61009$

Test,			
log sin a, b, c,	9.96820	9.98629	9.98284
log sin A, B, C,	9.94593	9.96403	9.96057
log modulus,	0.02227	0.02226	0.02227

2. Given, one angle = 128° 17′ 24″, another angle = 108° 39′ 52″, and the included side = 98° 40′ 32″.

Results,

$$\left.\begin{aligned} a &= 125^\circ\ 17'\ 22'' \\ b &= 99^\circ\ 51'\ 4'' \\ C &= 108^\circ\ 5'\ 15'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 305^\circ\ 17'\ 22'' \\ b &= 279^\circ\ 51'\ 4'' \\ C &= 251^\circ\ 54'\ 45'' \end{aligned}\right.$$

III.

302. Given two sides and the angle opposite one of them, a, b, and A.

The geometric construction is analogous to that in the corresponding case of plane triangles (p. 43). It is made on a globe as follows: Draw, for a base, the circumference of a great circle; draw an arc meeting this at a point **A** and forming with it an angle equal to the given angle A; on this arc measure off **A C** equal to the given side b; from the point **C** as pole, and with a polar radius equal to the given side a, draw a small circle cutting the base in two points $\mathbf{B}_1$ and $\mathbf{B}_2$; and finally draw the two arcs $\mathbf{C\,B}_1$ and $\mathbf{C\,B}_2$. Thus are formed two triangles $\mathbf{A\,C\,B}_1$ and $\mathbf{A\,C\,B}_2$, each of which contains, with the given parts, a separate value of each required part.

It may happen that all the parts of both triangles are each less than 180°; as in Fig. 70; or one of the triangles may have its base and the opposite angle (c and C) greater than 180°, as in Fig. 71; or both triangles may have the base and opposite angle greater than 180°, as might easily be shown by drawing another figure.

Since the given parts are limited not to exceed 180°, the modulus of both triangles is positive, and therefore c and C are either both less or both greater than 180°.

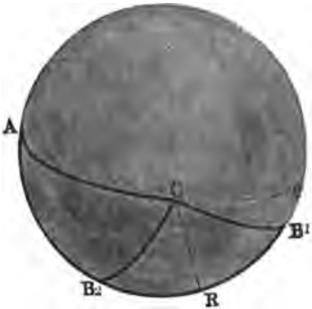

Fig. 70.

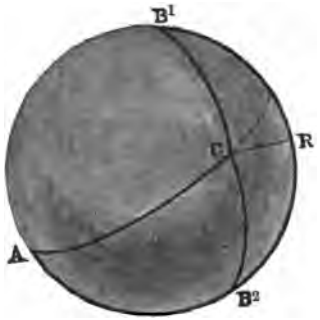

Fig. 71.

This construction fails to give triangles either when the polar radius a is too short to reach from **C** to the great circle c or when it is so long as to reach everywhere beyond it. Let **C R** be an arc less than 90° drawn from **C** perpendicular to c. Then, if a is shorter than **C R**, or longer than its supplement, the points $\mathbf{B}_1$ and $\mathbf{B}_2$ cannot be found, and there are no triangles; if a is equal to **C R** or to its supplement, the points $\mathbf{B}_1$ and $\mathbf{B}_2$ fall together at **R** or at a point diametrically opposite **R**, and the two triangles are one; if a is longer than **C R** but shorter than its supplement, the points $\mathbf{B}_1$ and $\mathbf{B}_2$ can be found, and there are two separate triangles. By Napier's second rule applied to the right triangle **A R C**,

$$\sin \mathrm{CR} = \sin b \sin A$$

which leads to a statement of the conditions of the special cases just described in the following form:

When a, b, and A are given the two triangles are

Both possible when	$\sin a > \sin b \sin A$
Identical when	$\sin a = \sin b \sin A$
Both impossible when	$\sin a < \sin b \sin A$

303. FIRST METHOD. By the fundamental equations [153], [152], and [155],

$$\sin B = \frac{\sin b \sin A}{\sin a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$\sin c = \frac{\cos^2 a - \cos^2 b}{\cos a \sin b \cos A - \sin a \cos b \cos B} \quad . \quad . \quad . \quad \text{(ii)}$$

$$\cos c = \frac{\sin b \cos b \cos A - \sin a \cos a \cos B}{\cos a \sin b \cos A - \sin a \cos b \cos B} \quad . \quad . \quad . \quad \text{(iii)}$$

$$\sin C = \frac{\cos^2 A - \cos^2 B}{\cos A \sin B \cos a - \sin A \cos B \cos b} \quad . \quad . \quad . \quad \text{(iv)}$$

$$\cos C = \frac{\sin A \cos A \cos b - \sin B \cos B \cos a}{\cos A \sin B \cos a - \sin A \cos B \cos b} \quad . \quad . \quad . \quad \text{(v)}$$

Formula (i) gives two values of B, which, when the given parts are limited not to exceed 180°, are both less than 180°, and

$$B_2 = 180° - B_1$$

In the first and second equations of [152] all the quantities are now known except sin c and cos c. These can be found by solving the two equations algebraically. The result of such a solution, in the general form, is (ii) and (iii). In the first two equations of [155], likewise, sin C and cos C are the only unknown quantities; and their values can be found by an algebraic solution; the result of which, in the general form, is (iv) and (v).

For each value of B substituted in the formulas (ii) and (iii) there results one and only one value of c, because these formulas determine the values of both the sine and the cosine of c, and when both these functions of an arc are determined, the arc itself can have but one value between 0° and 360°. Likewise, one and only one value of C results from the substitution of each value of B in (iv) and (v).

304. Examples.

1. Given, one side = 120°, another = 114°, and the angle opposite the former = 58°.

By the Table (p. 9),

	sin	cos
$a = 120°$	0.866	−0.500
$b = 114°$	0.914	−0.407
$A = 58°$	0.848	0.530

By substitution in the formulas,

$$\sin B = \frac{0.914 \times 0.848}{0.866} = 0.895$$

	sin	cos
$B_1 = 63° \, 30'$	0.895	0.446
$B_2 = 116° \, 30'$	0.895	−0.446

$$\sin c_1 = \frac{(-0.500)^2 - (-0.407)^2}{(-0.500)(0.914)(0.530) - (0.866)(-0.407)(0.446)} = -0.992$$

$$\cos c_1 = \frac{(0.914)(-0.407)(0.530) - (0.866)(-0.500)(0.446)}{(-0.500)(0.914)(0.530) - (0.866)(-0.407)(0.446)} = +0.047$$

$$c_1 = 272° \, 42' \, *$$

$$\sin C_1 = \frac{(0.530)^2 - (0.446)^2}{(0.530)(0.895)(-0.500) - (0.848)(0.446)(-0.407)} = -0.985$$

$$\cos C_1 = \frac{(0.848)(0.530)(-0.407) - (0.895)(0.446)(-0.500)}{(0.530)(0.895)(-0.500) - (0.848)(0.446)(-0.407)} = -0.200$$

$$C_1 = 258° \, 28'$$

By changing 0.446 to − 0.446 in the foregoing calculation the results are

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* The value of c_1 being near 270°, is more accurately found from its cosine than from its sine. For the *degree* of accuracy, compare results with those of the same example, page 226.

$\sin c_2 = -0.211$	$\sin C_2 = -0.210$
$\cos c_2 = +0.977$	$\cos C_2 = +0.978$
$c_2 = 347° 49'$	$C_2 = 347° 53'$ *

2. Given, one side $= 40° 16'$, another $= 47° 44'$, and the angle opposite the former $= 52° 30'$.

Results,

$$\left.\begin{array}{l} B = 65° 15' \\ c = 53° 30' \\ C = 80° \ 7' \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} B = 114° 45' \\ c = 14° 21' \\ C = 17° 39' \end{array}\right.$$

305. SECOND METHOD. The values of B having been found as in the First Method, c and C are found by substituting the values of B separately in the Gaussian Equations, which may be written as follows:

$$\text{(i)} \quad \log\cos\tfrac{1}{2}C - \log\cos\tfrac{1}{2}c = \log\sin\tfrac{1}{2}(A+B) - \log\cos\tfrac{1}{2}(a-b)$$
$$\text{(ii)} \quad \log\cos\tfrac{1}{2}C - \log\sin\tfrac{1}{2}c = \log\sin\tfrac{1}{2}(A-B) - \log\sin\tfrac{1}{2}(a-b)$$
$$\text{(iii)} \quad \log\sin\tfrac{1}{2}C - \log\cos\tfrac{1}{2}c = \log\cos\tfrac{1}{2}(A+B) - \log\cos\tfrac{1}{2}(a+b)$$
$$\text{(iv)} \quad \log\sin\tfrac{1}{2}C - \log\sin\tfrac{1}{2}c = \log\cos\tfrac{1}{2}(A-B) - \log\sin\tfrac{1}{2}(a+b)$$

306. Examples.

1. Given, one side $= 120°$, another $= 114°$, and the angle opposite the former $= 58°$.

$a = 120°$	log csc 0.06247	(1)
$b = 114°$	log sin 9.96073	(2)
$A = 58°$	log sin 9.92842	(3)
$B_1 = 63° 27' 17'' \pm 13''$	log sin 9.95162	(4) = (1) + (2) + (3)
$B_2 = 116° 32' 43''$		

---

* These values are found more accurately from their sines than from their cosines.

$\frac{1}{2}(A+B_1)=60^\circ\ 43'\ 38''$	log sin 9.94066	(5)	
	log cos 9.68928	(6)	
$\frac{1}{2}(A-B_1)=-(2^\circ\ 43'\ 38'')$	log sin 8.67746*n*	(7)	
	log cos 9.99951	(8)	
$\frac{1}{2}(a+b)=117^\circ$	log sin 9.94988	(9)	
	log cos 9.65705*n*	(10)	
$\frac{1}{2}(a-b)=3^\circ$	log sin 8.71880	(11)	
	log cos 9.99940	(12)	
(i) $\log\cos\frac{1}{2}C_1-\log\cos\frac{1}{2}c_1=$	9.94126	(13)	= (5)−(12)
(ii) $\log\cos\frac{1}{2}C_1-\log\sin\frac{1}{2}c_1=$	9.95866*n*	(14)	= (7)−(11)
(iii) $\log\sin\frac{1}{2}C_1-\log\cos\frac{1}{2}c_1=$	0.03223*n*	(15)	= (6)−(10)
(iv) $\log\sin\frac{1}{2}C_1-\log\sin\frac{1}{2}c_1=$	0.04963	(16)	= (8)−(9)
$\frac{1}{2}c_1=136^\circ\ 8'\ 50''.4$	log tan 9.98260*n*	(17)	=(13)−(14)
$c_1=272^\circ\ 17'\ 41''$	60*n*		=(15)−(16)
$\frac{1}{2}C_1=129^\circ\ 2'\ 33''.6$	log tan 0.09097*n*	(18)	=(15)−(13)
$C_1=258^\circ\ 5'\ 7''$	97*n*		=(16)−(14)
$\frac{1}{2}(A+B_2)=87^\circ\ 16'\ 22''$	log sin 9.99951	(8)	
	log cos 8.67746	(7)*n*	
$\frac{1}{2}(A-B_2)=-(29^\circ\ 16'\ 22'')$	log sin 9.68928*n*	(6)*n*	
	log cos 9.94066	(5)	
$\frac{1}{2}(a+b)=117^\circ$	log sin 9.94988	(9)	
	log cos 9.65705*n*	(10)	
$\frac{1}{2}(a-b)=3^\circ$	log sin 8.71880	(11)	
	log cos 9.99940	(12)	
(i) $\log\cos\frac{1}{2}C_2-\log\cos\frac{1}{2}c_2=$	0.00011	(19)	= (8) −(12)
(ii) $\log\cos\frac{1}{2}C_2-\log\sin\frac{1}{2}c_2=$	0.97048*n*	(20)	= (6)*n*−(11)
(iii) $\log\sin\frac{1}{2}C_2-\log\cos\frac{1}{2}c_2=$	9.02041*n*	(21)	= (7)*n*−(10)
(iv) $\log\sin\frac{1}{2}C_2-\log\sin\frac{1}{2}c_2=$	9.99078	(22)	= (5) − (9)
$\frac{1}{2}c_2=173^\circ\ 53'\ 21''$	log tan 9.02963*n*	(23)	=(19) −(20)
$c_2=347^\circ\ 46'\ 42''$	63*n*		=(21) −(22)
$\frac{1}{2}C_2=174^\circ\ 1'\ 5''$	log tan 9.02030*n*	(24)	=(21) −(19)
$C_2=348^\circ\ 2'\ 10''$	30*n*		=(22) −(20)

The course of the computation is easily followed by help of the reference numbers at the right hand. Numbers (1), (2), and (3) are taken out of the table, and (4), their sum, gives the two values of B. Numbers (5) to (12) are taken out of the table. From these are found, in the way indicated, numbers (13) to (16), which are the numerical values of the second members of equations (i), (ii), (iii), and (iv). Number (17) is log tan $\frac{1}{2}c_1$, found by subtracting the members of equation (ii) from those of (i) or those of (iv) from those of (iii). Number (18) is log tan $\frac{1}{2}C_1$, found by subtracting the members of (i) from those of (iii) or those of (ii) from those of (iv). Theoretically, the two ways of finding log tan $\frac{1}{2}c_1$, and the two ways of finding log tan $\frac{1}{2}C_1$, should give the same results, as they do in this example. What is to be done in case they do not is shown by the next example.

In continuing the computation with B_2 in place of B_1, there are no more logarithms to be taken out of the table. For, since

$$B_2 = 180° - B_1$$

we have

$$\begin{aligned}
\sin\tfrac{1}{2}(A+B_2) &= \cos\tfrac{1}{2}(A-B_1) = (8)\\
\cos\tfrac{1}{2}(A+B_2) &= -\sin\tfrac{1}{2}(A-B_1) = (7)n\\
\sin\tfrac{1}{2}(A-B_2) &= -\cos\tfrac{1}{2}(A+B_1) = (6)n\\
\cos\tfrac{1}{2}(A-B_2) &= \sin\tfrac{1}{2}(A+B_1) = (5)
\end{aligned}$$

which show that the first four logarithms of the second part of the computation are the same as the corresponding four of the first part, but arranged in a different order; for (5) and (8) change places, and (6) and (7) change places, the two latter, however, taking or dropping the n. If these changes be kept in mind, it is not even necessary to write the eight logarithms (5) to (12) again as was done above. The remaining steps are exactly like those of the first part already explained.

2. Given, one side = 70°, another = 100°, and the angle opposite the former = 69°.

$a = 70°$	log csc 0.02701		(1)
$b = 100°$	log sin 9.99335		(2)
$A = 69°$	log sin 9.97015		(3)
$B_1 = 78° 4' 0'' \pm 30''$	log sin 9.99051		(4)+(1)+(2)+(3)
$\frac{1}{2}(A+B_1) = 73° 32' 0''$ [7″]	log sin 9.98181		(5)
	log cos 9.45249	[244]	(6)
$\frac{1}{2}(A-B_1) = -(4° 32' 0'')$ [7″]	log sin 8.89784*n*	[803]	(7)
	log cos 9.99864		(8)
$\frac{1}{2}(a+b) = 85°$	log sin 9.99834		(9)
	log cos 8.94030		(10)
$\frac{1}{2}(a-b) = -15°$	log sin 9.41300*n*		(11)
	log cos 9.98494		(12)
(i) $\log\cos\frac{1}{2}C_1 - \log\cos\frac{1}{2}c_1 =$	9.99687		(13) = (5) − (12)
(ii) $\log\cos\frac{1}{2}C_1 - \log\sin\frac{1}{2}c_1 =$	9.48484	[503]	(14) = (7) − (11)
(iii) $\log\sin\frac{1}{2}C_1 - \log\cos\frac{1}{2}c_1 =$	0.51219	[214]	(15) = (6) − (10)
(iv) $\log\sin\frac{1}{2}C_1 - \log\sin\frac{1}{2}c_1 =$	0.00030		(16) = (8) − (9)
$\frac{1}{2}c_1 = 72° 53' 44''$	log tan 0.51203	[184]	(17) = (13) − (14)
	189	[184]	= (15) − (16)
$\frac{1}{2}C_1 = 73° 1' 21''$	log tan 0.51532	[527]	(18) = (15) − (13)
	546	[527]	= (16) − (14)
(i) $\log\cos\frac{1}{2}C_2 - \log\cos\frac{1}{2}c_2 =$	0.01370		(19) = (8) − (12)
(ii) $\log\cos\frac{1}{2}C_2 - \log\sin\frac{1}{2}c_2 =$	0.03944		(20) = (6)*n* − (11)
(iii) $\log\sin\frac{1}{2}C_2 - \log\cos\frac{1}{2}c_2 =$	9.95773		(21) = (7)*n* − (10)
(iv) $\log\sin\frac{1}{2}C_2 - \log\sin\frac{1}{2}c_2 =$	9.98347		(22) = (5) − (9)
$\frac{1}{2}c_2 = 43° 18' 11''.5$	log tan 9.97426		(23) = (19) − (20)
	26		= (21) − (22)
$\frac{1}{2}C_2 = 41° 19' 5''$	log tan 9.94403		(24) = (21) − (19)
	03		= (22) − (20)

$B_1 = 78° 4' 14''$ $B_2 = 101° 55' 46''$
$c_1 = 145° 47' 28''$ $c_2 = 86° 36' 23''$
$C_1 = 146° 2' 42''$ $C_2 = 82° 38' 10''$

The two values of number (17) differ by 14. The reason is that the value of B_1 taken out of the table is erroneous. Number (**4**) being the sum of three logarithms may be erroneous to the extent of $\pm \frac{3}{2}$ of a unit in the fifth decimal place. This $\pm \frac{3}{2}$ corresponds to a difference of $\pm 30''$ in the value of B_1. We are permitted, therefore, to make any needed alteration in the value of B_1 first found, if the alteration does not exceed $30''$. We first try the effect of increasing the value of B_1 by $2''$. This increases (**5**) and (**13**) by 0.07 in units of the fifth decimal place, diminishes (**6**) and (**15**) by 0.72, increases (**7**) and (**14**) by 2.65, and diminishes (**8**) and (**16**) by 0.02. The first value of (**17**), which is the difference between (**13**) and (**14**), is increased by 0.07 and diminished by 2.65; that is, on the whole, diminished by 2.58. The second value of (**17**), which is the difference between (**15**) and (**16**), is diminished by 0.72 and increased by 0.02; that is, on the whole, diminished by 0.70. The difference between the two values of (**17**) is diminished by 2.58 less 0.70, or 1.88. Thus an increase of $2''$ in the value of B_1 diminishes the difference between the two values of (**17**) by 1.88. To make this difference wholly disappear, we use $\frac{14}{1.88}$ times $2''$ or $14''$. This correction, $14''$, is within the limit of possible error in B_1, and is therefore to be accepted as giving a more accurate value of B_1 than that first obtained.

With this more accurate value of B_1 the computation is revised, the necessary alterations being written in brackets. When the values of $\frac{1}{2}\, c_1$ and $\frac{1}{2}\, C_1$ are taken out of the table, the figures in brackets are used instead of those first obtained. The second part of the computation is made with the corrected values.

3. Given, one side $= 55°$, another $= 114°$, and the angle opposite the former $= 58°$.

Results,

$$\left.\begin{aligned} B &= 71^\circ\ 2'\ 40'' \\ c &= 154^\circ\ 55'\ 26'' \\ C &= 153^\circ\ 58'\ 27'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} B &= 108^\circ\ 57'\ 20'' \\ c &= 105^\circ\ 8'\ 56'' \\ C &= 92^\circ\ 8'\ 51'' \end{aligned}\right.$$

NOTE. — The value of B first taken out is 71° 2′ 45″, which is corrected in the manner above explained.

4. Given, one side = 75°, another = 114°, and the angle opposite the former = 58°.

Results,

$$\left.\begin{aligned} B &= 53^\circ\ 19'\ 38'' \\ c &= 195^\circ\ 52'\ 24'' \\ C &= 193^\circ\ 53'\ 40'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} B &= 126^\circ\ 40'\ 22'' \\ c &= 64^\circ\ 12'\ 4'' \\ C &= 52^\circ\ 13'\ 42'' \end{aligned}\right.$$

307. THIRD METHOD. The two values of B, having been computed as in the Second Method, are separately substituted in Napier's Analogies [172], which may be written thus:

$$\text{(i)}\ \log\tan\tfrac{1}{2}c = \log\sin\tfrac{1}{2}(A+B) + \log\csc\tfrac{1}{2}(A-B) + \log\tan\tfrac{1}{2}(a-b)$$
$$\text{(ii)}\ \log\tan\tfrac{1}{2}c = \log\cos\tfrac{1}{2}(A+B) + \log\sec\tfrac{1}{2}(A-B) + \log\tan\tfrac{1}{2}(a+b)$$
$$\text{(iii)}\ \log\cot\tfrac{1}{2}C = \log\sin\tfrac{1}{2}(a+b) + \log\csc\tfrac{1}{2}(a-b) + \log\tan\tfrac{1}{2}(A-B)$$

308. Example.

Given, one side = 59° 32′, another = 75° 43′, and the angle opposite the former = 38° 25′.

a	$= 59^\circ\ 32'$	log csc 0.06453	
b	$= 75^\circ\ 43'$	log sin 9.98636	
A	$= 38^\circ\ 25'$	log sin 9.79335	
B_1	$= 44^\circ\ 19'\ 0'' \pm 7''$	log sin 9.84424	
		(i)	(ii)
$\frac{1}{2}(B_1+A)$	$= 41^\circ\ 22'\ 0''$ [1″]	log sin 9.82012	log cos 9.87535
$\frac{1}{2}(B_1-A)$	$= 2^\circ\ 57'\ 0''$ [1″]	log csc 1.28849 [45]	log sec 0.00058
$\frac{1}{2}(b+a)$	$= 67^\circ\ 37'\ 30''$		log tan 10.38546
$\frac{1}{2}(b-a)$	$= 8^\circ\ 5'\ 30''$	log tan 9.15282	
$\frac{1}{2}c_1$	$= 61^\circ\ 17'\ 12''$	log tan 10.26143 [39]	log tan 10.26139

$$\left.\begin{array}{l} B_1 = 44^\circ\ 19'\ \ 2'' \\ B_2 = 135^\circ\ 40'\ 58'' \end{array}\right\}\text{Corrected values.}$$

$\frac{1}{2}(B_1+A)$	$= 41^\circ\ 22'\ \ 1''$	(iii)	
$\frac{1}{2}(B_1-A)$	$= \ \ 2^\circ\ 57'\ \ 1''$	log tan 8.71212	
$\frac{1}{2}(b+a)$		log sin 9.96601	
$\frac{1}{2}(b-a)$		log csc 0.85153	
$\frac{1}{2}C_1$	$= 71^\circ\ 17'\ 41''$	log cot 9.52966	
		(i)	(iii)
$\frac{1}{2}(B_2+A)$	$= 87^\circ\ \ 2'\ 59''$	log sin 9.99942	
$\frac{1}{2}(B_2-A)$	$= 48^\circ\ 37'\ 59''$	log csc 0.12465	log tan 10.05523
$\frac{1}{2}(b+a)$			log sin 9.96601
$\frac{1}{2}(b-a)$		log tan 9.15282	log csc 0.85153
$\frac{1}{2}c_2$	$= 10^\circ\ 42'\ 47''$	log tan 9.27689	
$\frac{1}{2}C_2$	$= \ \ 7^\circ\ 38'\ \ 4''$		log cot 10.87277

Results,

$$\left.\begin{array}{l} B = \ \ 44^\circ\ 19'\ \ 2'' \\ c = 122^\circ\ 34'\ 24'' \\ C = 142^\circ\ 35'\ 22'' \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} B = 135^\circ\ 40'\ 58'' \\ c = \ \ 21^\circ\ 25'\ 34'' \\ C = \ \ 15^\circ\ 16'\ \ 8'' \end{array}\right.$$

The value of B_1 first taken out may be erroneous to the extent of ± 7″. The two values of log tan $\frac{1}{2}c$ computed by (i) and (ii) can be made to agree by adding 2″ to this value of B_1, thus making it 44° 19′ 2″. The alterations in the logarithms are shown in the brackets. With the corrected values of B_1 and B_2 the computation is finished.

309. FOURTH METHOD. By drawing a great-circle arc C R from C perpendicular to the opposite side, two right triangles C R A and C R B are formed which may be solved as follows:

To find B.

$$\sin B = \csc a \sin b \sin A \quad . \ . \ . \ . \ . \quad \text{(i)}$$

From the second and fourth equations of [173] we learn that in a spherical right triangle, none of whose parts exceeds 180°, an oblique angle and the side opposite to it are either both less than 90° or both greater than 90°. This requires, when the perpendicular falls within the required triangle (Figs. 70 and 71), that both the perpendicular and the angle B be in the same quadrant (first or second) with the given angle A. That value of B which is in the same quadrant with A is designated by B_1, the other by B_2.

To find c.

In the right triangle **C R A**, by Napier's first rule,

$$\tan \mathrm{AR} = \tan b \cos A \quad . \; . \; . \; . \; . \; . \quad \text{(ii)}$$

Napier's second rule applied to the two right triangles **C R A** and **C R B** gives the equations

$$\cos b = \cos \mathrm{AR} \cos \mathrm{CR}$$
$$\cos a = \cos \mathrm{RB} \cos \mathrm{CR}$$

whence, by eliminating cos **C R**,

$$\cos \mathrm{RB} = \cos \mathrm{AR} \cos a \sec b \quad . \; . \; . \quad \text{(iii)}$$

$$\left.\begin{array}{l} c_1 = \mathrm{AR} + \mathrm{RB} \\ c_2 = \mathrm{AR} - \mathrm{RB} \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(iv)}$$

To find C.

In the right triangle **C R A**, by Napier's first rule,

$$\tan \mathrm{ACR} = \sec b \cot A \quad . \; . \; . \; . \; . \; . \; . \quad \text{(v)}$$

The same rule applied to the two right triangles **C R A** and **C R B** gives the equations

$$\cos \mathrm{ACR} = \cot b \tan \mathrm{CR}$$
$$\cos \mathrm{RCB} = \cot a \tan \mathrm{CR}$$

whence, by eliminating tan **C R**,

$$\cos \mathrm{RCB} = \cos \mathrm{ACR} \cot a \tan b \quad . \; . \; . \quad \text{(vi)}$$

$$\left.\begin{array}{l} C_1 = \mathrm{ACR} + \mathrm{RCB} \\ C_2 = \mathrm{ACR} - \mathrm{RCB} \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(vii)}$$

310. **Examples.**

1. Given, one side = 120°, another = 114°, and the angle opposite the former = 58°.

(i)

$$\begin{array}{ll} a = 120^\circ & \log\csc 0.06247 \\ b = 114^\circ & \log\sin 9.96073 \\ A = 58^\circ & \log\sin \underline{9.92842} \\ B_1 = 63^\circ\ 27'\ 17'' & \log\sin 9.95162 \\ B_2 = 116^\circ\ 32'\ 43'' & \end{array}$$

(ii)

$$\begin{array}{ll} \log\tan b & = 10.35142n \\ \log\cos A & = \underline{\ 9.72421\ } \\ \log\tan \mathrm{AR} & = 10.07563n \end{array}$$

$$\mathrm{AR} = 130^\circ\ 2'\ 10''$$

(iii)

$$\begin{array}{ll} \log\cos \mathrm{AR} & = 9.80839 \\ \log\cos a & = 9.69897n \\ \log\sec b & = \underline{0.39069n} \\ \log\cos \mathrm{RB} & = 9.89805n \end{array}$$

$$\mathrm{RB} = 142^\circ\ 15'\ 27''$$

(iv) $c_1 = 272^\circ\ 17'\ 37''$

$c_2 = 347^\circ\ 46'\ 43''$

(v)

$$\begin{array}{ll} \log\sec b & = 0.39069n \\ \log\cot A & = \underline{\ 9.79579\ } \\ \log\tan \mathrm{ACR} & = 10.18648n \end{array}$$

$$\mathrm{ACR} = 123^\circ\ 3'\ 38''$$

(vi)

$$\begin{array}{ll} \log\cos \mathrm{ACR} & = 9.73682n \\ \log\cot a & = 9.76144n \\ \log\tan b & = \underline{10.35142n} \\ \log\cos \mathrm{RCB} & = 9.84968n \end{array}$$

$$\mathrm{RCB} = 135^\circ\ 1'\ 32''$$

(vii) $C_1 = 258^\circ\ 5'\ 10''$

$C_2 = 348^\circ\ 2'\ 6''$

NOTE. — Compare the results with those of the same example, Art. 306.

2. Given one side = 123° 46′ 22″, another = 78° 33′ 38″ and the angle opposite the former = 133° 21′ 12″.

Results,

$$\begin{array}{ll} B_1 = 120^\circ\ 58'\ 30'' & B_2 = 59^\circ\ 1'\ 30'' \\ c_1 = 144^\circ\ 0'\ 24'' & c_2 = 68^\circ\ 50'\ 24'' \\ C_1 = 149^\circ\ 3'\ 50'' & C_2 = 54^\circ\ 39'\ 54'' \end{array}$$

IV.

311. Given two angles and the side opposite one of them, A, B, and a.

As a preliminary step in the geometric construction, draw a right triangle on the globe with the given side *a* for a hypothenuse and the given angle *B* for an oblique angle. The side of this right triangle which is opposite the angle *B* is equal to the perpendicular **C R** of the required triangle.

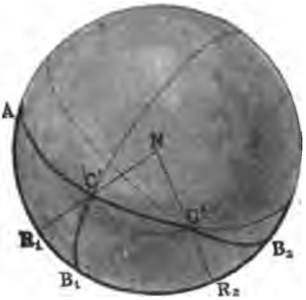


Fig. 72.

Now, from any point **N** as pole draw a great and a small circle, the latter with a polar radius equal to the difference between **C R** and a quadrant. The base of the required triangle being placed in the circumference of this great circle, the vertex **C** falls in that of the small circle. At any point **A** in the circumference of the great circle construct an angle equal to the given angle *A*, and extend its side until it cuts the circumference of the small circle in two points $\mathbf{C}_1$ and $\mathbf{C}_2$. These are vertices of two triangles, each of which has the angle *A* as one of its parts. From these two vertices draw perpendiculars $\mathbf{C}_1\mathbf{R}_1$ and $\mathbf{C}_2\mathbf{R}_2$ to the base and complete the right triangles $\mathbf{C}_1\mathbf{R}_1\mathbf{B}_1$ and $\mathbf{C}_2\mathbf{R}_2\mathbf{B}_2$, each being equal to the preliminary right triangle, and being so placed as to bring the given angle *B* within the required triangle. See Figs. 72, 73, and 74.

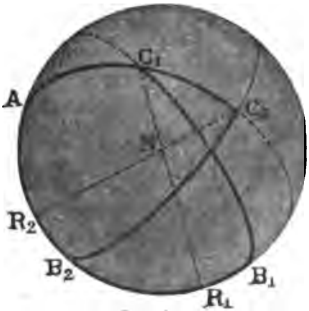


Fig. 73.

Thus are formed two triangles, each containing with the given parts a separate value of each required part.

It may happen that all the parts of both triangles are each less than 180°, as in Figs. 72 and 73; or one of the triangles may have its base and opposite angle (c and C) greater than 180°, Fig. 74.

This construction fails to give triangles when the angle A is either so small or so large that its side on being extended does not cut the circumference of the small circle. Then the points $\mathbf{C}_1$ and $\mathbf{C}_2$ cannot be found. Both triangles are one and the same when the angle A is such that its side on being extended touches the circumference of the small circle in one point.

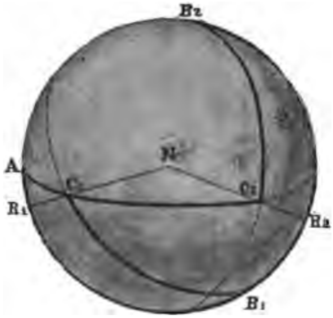


Fig. 74.

The angle A is measured by an arc of a great circle drawn from its vertex as a pole and included between its sides. Such an arc would extend from a point half-way between $\mathbf{C}_1$ and $\mathbf{C}_2$ to a point half-way between $\mathbf{R}_1$ and $\mathbf{R}_2$. Now when there are two triangles this arc measuring the angle A is greater than **CR** when **CR** is acute (Fig. 72) or less than **CR** when **CR** is obtuse (Fig. 73); that is, in either case, $\sin A > \sin \mathbf{CR}$. When both triangles are one, the arc measuring the angle A is equal to **CR** or to its supplement; that is, in either case, $\sin A = \sin \mathbf{CR}$. When there are no triangles, $\sin A < \sin \mathbf{CR}$.

By Napier's second rule applied to the right triangle **CRB**,

$$\sin \mathbf{CR} = \sin B \sin a$$

which leads to a statement of the conditions of the special cases just described in the following form:

When A, B, and a are given, the triangles are

Both possible when $\sin A > \sin B \sin a$
Identical when $\sin A = \sin B \sin a$
Both impossible when $\sin A < \sin B \sin a$

312. First Method. By the fundamental equations [153], [152], and [155],

$$\sin b = \frac{\sin B \sin a}{\sin A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$\sin c = \frac{\cos^2 a - \cos^2 b}{\cos a \sin b \cos A - \sin a \cos b \cos B} \quad . \quad . \quad . \quad \text{(ii)}$$

$$\cos c = \frac{\sin b \cos b \cos A - \sin a \cos a \cos B}{\cos a \sin b \cos A - \sin a \cos b \cos B} \quad . \quad . \quad . \quad \text{(iii)}$$

$$\sin C = \frac{\cos^2 A - \cos^2 B}{\cos A \sin B \cos a - \sin A \cos B \cos b} \quad . \quad . \quad \text{(iv)}$$

$$\cos C = \frac{\sin A \cos A \cos b - \sin B \cos B \cos a}{\cos A \sin B \cos a - \sin A \cos B \cos b} \quad . \quad . \quad . \quad \text{(v)}$$

Formula (i) gives supplementary values of b, each less than 180°. The other formulas are identical with those given in Art. 303, and are to be used in the same way. For each value of b substituted in them, (ii) and (iii) give one and only one value of c, while (iv) and (v) give one and only one value of C.

313. Examples.

1. Given, one angle $= 64°$, another $= 72°$, and the side opposite the former $= 57°$.

By the Table (p. 9),

	sin	cos
$A=64°$	0.899	0.438
$B=72°$	0.951	0.309
$a=57°$	0.839	0.545

$$\sin b=\frac{0.951\times 0.839}{0.899}=0.8875$$

	sin	cos
$b_1=62°\ 34'$	0.8875	0.461
$b_2=117°\ 26'$	0.8875	-0.461

$$\sin c_1=\frac{(0.545)^2-(0.461)^2}{(0.545)(0.8875)(0.438)-(0.839)(0.461)(0.309)}=0.915$$

$$\cos c_1=\frac{(0.8875)(0.461)(0.438)-(0.839)(0.545)(0.309)}{(0.545)(0.8875)(0.438)-(0.839)(0.461)(0.309)}=0.411$$

$c_1=65°\ 45'$, found from its cosine.

$$\sin C_1=\frac{(0.438)^2-(0.309)^2}{(0.438)(0.951)(0.545)-(0.899)(0.309)(0.461)}=0.974$$

$$\cos C_1=\frac{(0.899)(0.438)(0.461)-(0.951)(0.309)(0.545)}{(0.438)(0.951)(0.545)-(0.899)(0.309)(0.461)}=0.216$$

$C_1=77°\ 32'$, found from its cosine.

By changing 0.461 to -0.461 in the foregoing calculation the results are:

$\sin c_2=0.255$ $\qquad$ $\sin C_2=0.271$

$\cos c_2=-0.967$ $\qquad$ $\cos C_2=-0.962$

$c_2=165°\ 14'$ $\qquad$ $C_2=164°\ 18'$

314. SECOND METHOD. The values of b having been found as in the First Method, c and C are found by substituting the values of b separately in the Gaussian Equations, which for this purpose are written just as they were in Art. 305.

315. Examples.

1. Given, one angle $=75°\ 18'$, another $=57°\ 24'$, and the side opposite the former $=131°\ 14'$.

$A = 75^\circ\ 18'$	log csc	0.01445	**(1)**
$B = 57^\circ\ 24'$	log sin	9.92555	**(2)**
$a = 131^\circ\ 14'$	log sin	9.87624	**(3)**
$b_1 = 40^\circ\ 55'\ 8''$	log sin	9.81624	**(4)=(1)+(2)+(3)**
$b_2 = 139^\circ\ 4'\ 52''$			
$\frac{1}{2}(A+B) = 66^\circ\ 21'$	log sin	9.96190	**(5)**
	log cos	9.60331	**(6)**
$\frac{1}{2}(A-B) = 8^\circ\ 57'$	log sin	9.19193	**(7)**
	log cos	9.99468	**(8)**
$\frac{1}{2}(a+b_1) = 86^\circ\ 4'\ 34''$	log sin	9.99898	**(9)**
	log cos	8.83526	**(10)**
$\frac{1}{2}(a-b_1) = 45^\circ\ 9'\ 26''$	log sin	9.85067	**(11)**
	log cos	9.84829	**(12)**

(i) $\log\cos\frac{1}{2}C_1 - \log\cos\frac{1}{2}c_1 =$	0.11361	**(13)= (5)−(12)**
(ii) $\log\sin\frac{1}{2}C_1 - \log\sin\frac{1}{2}c_1 =$	9.34126	**(14)= (7)−(11)**
(iii) $\log\sin\frac{1}{2}C_1 - \log\cos\frac{1}{2}c_1 =$	0.76805	**(15)= (6)−(10)**
(iv) $\log\sin\frac{1}{2}C_1 - \log\sin\frac{1}{2}c_1 =$	9.99570	**(16)= (8)− (9)**

$\frac{1}{2}c_1 = 80^\circ\ 24'\ 46''$	log tan	10.77235	**(17)=(13)−(14)**
		35	**=(15)−(16)**
$\frac{1}{2}C_1 = 77^\circ\ 30'\ 20''$	log tan	10.65444	**(18)=(15)−(13)**
		44	**=(16)−(14)**

(i) $\log\cos\frac{1}{2}C_2 - \log\cos\frac{1}{2}c_2 =$	9.96292	**(19)= (5)− (9)**
(ii) $\log\cos\frac{1}{2}C_2 - \log\sin\frac{1}{2}c_2 =$	0.35667*n*	**(20)= (7)−(10)***n*
(iii) $\log\sin\frac{1}{2}C_2 - \log\cos\frac{1}{2}c_2 =$	9.75264*n*	**(21)= (6)−(11)***n*
(iv) $\log\sin\frac{1}{2}C_2 - \log\sin\frac{1}{2}c_2 =$	0.14639	**(22)= (8)−(12)**

$\frac{1}{2}c_2 = 158^\circ\ 0'\ 27''$	log tan	9.60625*n*	**(23)=(19)−(20)**
		25	**=(21)−(22)**
$\frac{1}{2}C_2 = 148^\circ\ 21'\ 32''$	log tan	9.78972*n*	**(24)=(21)−(19)**
		72	**=(22)−(20)**

$c_1 = 160^\circ\ 49'\ 32''$
$C_1 = 155^\circ\ 0'\ 40''$

$c_2 = 316^\circ\ 0'\ 54''$
$C_2 = 296^\circ\ 43'\ 4''$

The course of the computation is in all particulars like that of Ex. 1, Art. 306, except that in the second part the changes there made in logarithms (5) to (8) are here made in (9) to (12) inclusive. That is, when b_2 takes the place of b_1, (9) and (12) change places, and (10) and (11) change places, the two latter adding or dropping the n. The shape of the triangles is represented in Fig. 74.

2. Given, one angle = 64°, another = 72°, and the side opposite the former = 57°.

Results,

$$\left.\begin{array}{l} b = 62^\circ\ 33'\ 10'' \\ c = 65^\circ\ 36'\ 45'' \\ C = 77^\circ\ 26'\ 16'' \end{array}\right\}\quad \text{or} \quad \left\{\begin{array}{l} b = 117^\circ\ 26'\ 50'' \\ c = 165^\circ\ 16'\ 59'' \\ C = 164^\circ\ 12'\ \ 5'' \end{array}\right.$$

Note. — The shape of the triangles is represented in Fig. 72.

3. Given, one angle = 124° 22′, another = 109° 16′, and the side opposite the former = 121° 47′.

$A = 124^\circ\ 22'$	log csc	0.08331		(1)
$B = 109^\circ\ 16'$	log sin	9.97497		(2)
$a = 121^\circ\ 47'$	log sin	9.92944		(3)
$b_1 = 76^\circ\ 26'\ 20'' \pm 30''$	log sin	9.98772		(4) = (1) + (2) + (3)
$\frac{1}{2}(A+B) = 116^\circ\ 49'$	log sin	9.95059		(5)
	log cos	$9.65431n$		(6)
$\frac{1}{2}(A-B) = 7^\circ\ 33'$	log sin	9.11857		(7)
	log cos	9.99622		(8)
$\frac{1}{2}(a+b_1) = 99^\circ\ 6'\ 40''$ [44″]	log sin	9.99449		(9)
	log cos	$9.19962n$	[67]	(10)
$\frac{1}{2}(a-b_1) = 22^\circ\ 40'\ 20''$ [16″]	log sin	9.58598	[96]	(11)
	log cos	9.96507		(12)
		9.98552		(13) = (5) − (12)
		9.53259	[61]	(14) = (7) − (11)
		0.45469	[64]	(15) = (6) − (10)
		0.00173		(16) = (8) − (9)
$\frac{1}{2}c_1 = 70^\circ\ 35'\ 7''.3$	log tan	10.45293	[91]	(17) = (13) − (14)
		96	[91]	= (15) − (16)

$\frac{1}{2} C_1 = 71° 14' 45''.7$	log tan	10.46917	[12]	(18) = (15) − (13)
		14	[12]	= (16) − (14)
		9.95610		(19) = (5) − (9)
		9.91890		(20) = (7) − (10)*n*
		0.06835		(21) = (6) − (11)*n*
		0.03115		(22) = (8) − (12)
$\frac{1}{2} c_2 = 47° 27' 2''.4$	log tan	10.03720		(23) = (19) − (20)
		20		= (21) − (22)
$\frac{1}{2} C_2 = 52° 19' 24''.4$	log tan	10.11225		(24) = (21) − (19)
		25		= (22) − (20)

$$\begin{array}{ll} b_1 = 76° 26' 28'' & b_2 = 103° 33' 32'' \\ c_1 = 141° 10' 15'' & c_2 = 94° 54' 5'' \\ C_1 = 142° 29' 31'' & C_2 = 104° 38' 49'' \end{array}$$

Note. — The corrections in brackets are made in the manner and for reasons explained under Ex. 2, Art. 306. The general shape of these triangles is represented in Fig. 73.

4. Given, one angle = 63° 40′, another = 72° 18′, and the side opposite the former = 68° 50′.

Results,

$$\left.\begin{array}{l} b = 82° 24' 58'' \\ c = 111° 25' 56'' \\ C = 116° 32' 34'' \end{array}\right\} \text{ or } \left\{\begin{array}{l} b = 97° 35' 2'' \\ c = 144° 50' 44'' \\ C = 146° 24' 8'' \end{array}\right.$$

5. Given, one angle = 125° 32′, another = 114° 45′, and the side opposite the former = 129° 14′.

Results,

$$\left.\begin{array}{l} b = 59° 48' 50'' \\ c = 162° 10' 14'' \\ C = 161° 14' 7'' \end{array}\right\} \text{ or } \left\{\begin{array}{l} b = 120° 11' 10'' \\ c = 72° 7' 9'' \\ C = 90° 56' 28'' \end{array}\right.$$

6. Given, one angle = 109° 20′, another = 128° 38′, and the side opposite the former = 117° 22′.

Results,

$$\left.\begin{array}{l} b = 47^\circ\, 19'\, 25'' \\ c = 210^\circ\, 35'\, 15'' \\ C = 212^\circ\, 43'\, 46'' \end{array}\right\} \text{ or } \left\{\begin{array}{l} b = 132^\circ\, 40'\, 35'' \\ c = 70^\circ\, 5'\, 27'' \\ C = 87^\circ\, 27'\, 34'' \end{array}\right.$$

7. Given, one angle = 63° 28′, another = 134° 42′, and the side opposite the former = 73° 38′.

Results,

$$\left.\begin{array}{l} b = 49^\circ\, 39'\, 58'' \\ c = 320^\circ\, 24'\, 44'' \\ C = 323^\circ\, 32'\, 34'' \end{array}\right\} \text{ or } \left\{\begin{array}{l} b = 130^\circ\, 20'\, 2'' \\ c = 84^\circ\, 54'\, 46'' \\ C = 68^\circ\, 14'\, 46'' \end{array}\right.$$

316. THIRD METHOD. The two values of b, having been computed as in the Second Method, are separately substituted in Napier's Analogies, which are written in the same form as in Art. 307. The method of computation is that of the Example in Art. 308.

317. FOURTH METHOD. By drawing a great-circle arc CR from C perpendicular to the opposite side two right triangles are formed, which may be solved as follows:

To find b.

$$\sin b = \csc A \sin B \sin a \quad . \; . \; . \; . \; . \quad \text{(i)}$$

To find c.

In the right triangle BRC, by Napier's first rule,

$$\cot \mathrm{BCR} = \tan B \cos a \quad . \; . \; . \; . \; . \; . \quad \text{(ii)}$$

Napier's second rule applied to the two right triangles CRA and CRB, gives the equations

$$\cos A = \sin \mathrm{RCA} \cos \mathrm{CR}$$
$$\cos B = \sin \mathrm{BCR} \cos \mathrm{CR}$$

whence, by eliminating cos CR,

$$\sin \mathrm{RCA} = \sin \mathrm{BCR} \cos A \sec B \quad . \; . \; . \quad \text{(iii)}$$

which gives supplementary values of RCA.

Then

$$\left.\begin{array}{l} C_1 = \mathrm{B\,C\,R} + \mathrm{R\,C\,A}_1 \\ C_2 = \mathrm{B\,C\,R} + \mathrm{R\,C\,A}_2 \end{array}\right\} \quad . \ . \ . \ . \ . \ . \quad \text{(iv)}$$

To find *c*.

In the right triangle **R C B**, by Napier's first rule,

$$\cot \mathrm{B\,R} = \sec B \cot a \quad . \ . \ . \ . \ . \ . \quad \text{(v)}$$

The same rule applied to the right triangles **C R A** and **C R B** gives the equations

$$\sin \mathrm{R\,A} = \cot A \tan \mathrm{C\,R}$$
$$\sin \mathrm{B\,R} = \cot B \tan \mathrm{C\,R}$$

whence by eliminating tan **C R**,

$$\sin \mathrm{R\,A} = \sin \mathrm{B\,R} \cot A \tan B \quad . \ . \ . \ . \ . \ . \quad \text{(vi)}$$

which gives supplementary values of **R A**.

Then

$$\left.\begin{array}{l} c_1 = \mathrm{B\,R} + \mathrm{R\,A}_1 \\ c_2 = \mathrm{B\,R} + \mathrm{R\,A}_2 \end{array}\right\} \quad . \ . \ . \ . \ . \ . \ . \quad \text{(vii)}$$

NOTE.—If (iii) and (vi) give negative values of sin **R C A** and sin **R A**, then

$$\mathrm{R\,C\,A}_1 + \mathrm{R\,C\,A}_2 = 540^\circ \text{ and } \mathrm{R\,A}_1 + \mathrm{R\,A}_2 = 540^\circ.$$

318. Examples.

1. Given, one angle $= 78^\circ\ 14'\ 23''$, another $= 82^\circ\ 19'\ 15''$, and the side opposite the former $= 62^\circ\ 28'\ 17''$.

	(i)
$A = 78^\circ\ 14'\ 23''$	log csc 0.00921
$B = 82^\circ\ 19'\ 15''$	log sin 9.99609
$a = 62^\circ\ 28'\ 17''$	log sin $\underline{9.94782}$
$b_1 = 63^\circ\ 51'\ 17''$	log sin 9.95312
$b_2 = 116^\circ\ 8'\ 43''$	

(ii)

$\log\tan B = 10.87020$
$\log\cos a = \underline{9.66482}$
$\log\cot \mathrm{BCR} = 10.53502$
$\mathrm{BCR} = 16^\circ\, 15'\, 49''$

(iii)

$\log\sin \mathrm{BCR} = 9.44725$
$\log\cos A = 9.30924$
$\log\sec B = \underline{0.87411}$
$\log\sin \mathrm{RCA} = 9.63060$

$\mathrm{RCA}_1 = 25^\circ\, 17'\, 18''$
$\mathrm{RCA}_2 = 154^\circ\, 42'\, 42''$

(iv)

$C_1 = 41^\circ\, 33'\, 7''$
$C_2 = 170^\circ\, 58'\, 31''$

(v)

$\log\sec B = 0.87411$
$\log\cot a = \underline{9.71701}$
$\log\cot \mathrm{BR} = 10.59112$
$\mathrm{BR} = 14^\circ\, 22'\, 47''$

(vi)

$\log\sin \mathrm{BR} = 9.39506$
$\log\cot A = 9.31845$
$\log\tan B = \underline{10.87020}$
$\log\sin \mathrm{RA} = 9.58371$

$\mathrm{RA}_1 = 22^\circ\, 32'\, 52''$
$\mathrm{RA}_2 = 157^\circ\, 27'\, 8''$

(vii)

$c_1 = 36^\circ\, 55'\, 39''$
$c_2 = 171^\circ\, 49'\, 55''$

Note. — These triangles are of the shape represented in Fig. 72.

2. Given, one angle $= 63^\circ\, 28'$, another $= 134^\circ\, 42'$, and the side opposite the former $= 73^\circ\, 38'$.

$A = 63^\circ\, 28'$
$B = 134^\circ\, 42'$
$a = 73^\circ\, 38'$
$b_1 = 49^\circ\, 40'\, 0''$
$b_2 = 130^\circ\, 20'\, 0''$

(i)

$\log\csc = 0.04833$
$\log\sin = 9.85175$
$\log\sin = \underline{9.98204}$
$\log\sin = 9.88212$

(ii)

$\log\tan B = 10.00455n$
$\log\cos a = \underline{9.44992}$
$\log\cot \mathrm{BCR} = 9.45447n$
$\mathrm{BCR} = 105^\circ\, 53'\, 40''$

(v)

$\log\sec B = 0.15280n$
$\log\cot a = \underline{9.46788}$
$\log\cot \mathrm{BR} = 9.62068n$
$\mathrm{BR} = 112^\circ\, 39'\, 42''$

(iii)

$$\begin{aligned} \log \sin \mathrm{BCR} &= 9.98307 \\ \log \cos A &= 9.65003 \\ \log \sec B &= \underline{0.15280n} \\ \log \sin \mathrm{RCA} &= 9.78590n \end{aligned}$$

$$\begin{aligned} \mathrm{RCA}_1 &= 217^\circ\ 38'\ 52'' \\ \mathrm{RCA}_2 &= 322^\circ\ 21'\ \ 8'' \end{aligned}$$

(iv)

$$\begin{aligned} C_1 &= 323^\circ\ 32'\ 32'' \\ C_2 &= 68^\circ\ 14'\ 48'' \end{aligned}$$

(vi)

$$\begin{aligned} \log \sin \mathrm{BR} &= 9.96511 \\ \log \cot A &= 9.69837 \\ \log \tan B &= \underline{10.00455n} \\ \log \sin \mathrm{RA} &= 9.66803n \end{aligned}$$

$$\begin{aligned} \mathrm{RA}_1 &= 207^\circ\ 45'\ 0'' \\ \mathrm{RA}_2 &= 332^\circ\ 15'\ 0'' \end{aligned}$$

(vii)

$$\begin{aligned} c_1 &= 320^\circ\ 24'\ 42'' \\ c_2 &= 84^\circ\ 54'\ 42'' \end{aligned}$$

Note.—The general shape of these triangles will be represented by Fig. 74 if certain changes be made, namely, by drawing the perpendiculars from C_1 and C_2 through N, and placing R_1 and R_2 also B_1 and B_2 at points diametrically opposite where they now stand.

3. Given, one angle = 125° 32′, another = 114° 45′, and the side opposite the former = 129° 14′.

Results,

$$\left.\begin{aligned} b &= 59^\circ\ 48'\ 38'' \\ c &= 162^\circ\ 10'\ 14'' \\ C &= 161^\circ\ 14'\ \ 0'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} b &= 120^\circ\ 11'\ 22'' \\ c &= 72^\circ\ \ 7'\ 10'' \\ C &= 90^\circ\ 56'\ 32'' \end{aligned}\right.$$

Note.—Under Ex. 5, Art. 315, we learn that the value of b here found is erroneous, and should be corrected by adding 12″; but the method of solution here used affords no means of finding such a correction. Fig. 73 gives the shape of the triangles.

4. Given, one angle = 124° 22′, another = 109° 16′, and the side opposite the former = 121° 47′.

Results,

$$\left.\begin{aligned} b &= 76^\circ\ 26'\ 20'' \\ c &= 141^\circ\ 10'\ 12'' \\ C &= 142^\circ\ 29'\ 35'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} b &= 103^\circ\ 33'\ 40'' \\ c &= 94^\circ\ 54'\ 12'' \\ C &= 104^\circ\ 38'\ 47'' \end{aligned}\right.$$

V.

319. Given, the three sides, a, b, and c.

The geometric construction is made by drawing a great-circle arc equal to a; from one end of this as a pole with a polar radius equal to b, and from the other end with a polar radius equal to c drawing arcs to cross in a point; and joining this point to the ends of a by great-circle arcs. The angles of the triangle thus formed are either the three interior angles or the three exterior angles, the former being each less than 180°, and the latter being each greater than 180°.

320. First Method. By the fundamental equations [152],

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$\cos B=\frac{\cos b-\cos c\cos a}{\sin c\sin a}$$

$$\cos C=\frac{\cos c-\cos a\cos b}{\sin a\sin b}$$

Each of these equations gives two values of an angle, one less and one greater than 180°, and such that

$$A_1+A_2=B_1+B_2=C_1+C_2=360^\circ$$

321. Example.

Given, one side = 76°, another = 79°, and another = 83°. By the Table (p. 9).

	sin	cos
$a=76^\circ$	0.970	0.242
$b=79^\circ$	0.982	0.191
$c=83^\circ$	0.993	0.122

By substitution in the equations,

$$\cos A=\frac{(0.242)-(0.191)(0.122)}{(0.982)(0.993)}=0.224$$

$$\cos B=\frac{(0.191)-(0.122)(0.242)}{(0.993)(0.970)}=0.168$$

$$\cos C=\frac{(0.122)-(0.242)(0.191)}{(0.970)(0.982)}=0.080$$

$$\begin{array}{lll} A_1=\ 77^\circ\ 4' & B_1=\ 80^\circ\ 20' & C_1=\ 85^\circ\ 25' \\ A_2=282^\circ\ 56' & B_2=279^\circ\ 40' & C_2=274^\circ\ 35' \end{array}$$

322. SECOND METHOD. The halves of the angles can be computed from their sines [161], or from their cosines [163], or from their tangents [164]. The most convenient form of computation from the tangents is based on [174] and [175], namely

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan\tfrac{1}{2}A=\frac{\tan r}{\sin(s-a)},\ \tan\tfrac{1}{2}B=\frac{\tan r}{(\sin s-b)},\ \tan\tfrac{1}{2}C=\frac{\tan r}{\sin(s-c)}.$$

The right-hand members of these equations being square roots are to be taken as both positive and negative. Hence each half angle has two values, which doubled are the two values of the whole angle.

323. Examples.

1. Given the same values as in Art. 321.

$$\begin{array}{rl} a= & 76^\circ \\ b= & 79^\circ \\ c= & \underline{83^\circ} \\ & 238^\circ \end{array}$$

$s = 119°$	log csc	0.05818	(1)
$s-a = 43°$	log sin	9.83378	(2)
$s-b = 40°$	log sin	9.80807	(3)
$s-c = 36°$	log sin	9.76922	(4)
		19.46925	(5)
	log tan $r=$	9.73462	(6) = ½ (5)
$\frac{1}{2}A_1 = 38° 30' 55''$	log tan	9.90084	(7) = (6) − (2)
$\frac{1}{2}B_1 = 40° 10' 41''$	log tan	9.92655	(8(= (6) − (3)
$\frac{1}{2}C_1 = 42° 43' 12''$	log tan	9.96540	(9) = (6) − (4)

Taking the tangents as negative,

$\frac{1}{2}A_2 = 141° 29' 5''$
$\frac{1}{2}B_2 = 139° 49' 19''$
$\frac{1}{2}C_2 = 137° 16' 48''$

$A_1 = 77° 1' 50''$ $A_2 = 282° 58' 10''$
$B_1 = 80° 21' 22''$ $B_2 = 279° 38' 38''$
$C_1 = 85° 26' 24''$ $C_2 = 274° 33' 36''$

2. Given, the three sides 109° 33′, 121° 44′, and 53° 35′.

Results,

$$\left.\begin{array}{l} A = 91° 52' 31'' \\ B = 115° 34' 15'' \\ C = 58° 35' 38'' \end{array}\right\} \text{ or } \left\{\begin{array}{l} A = 268° 7' 29'' \\ B = 244° 25' 45'' \\ C = 301° 24' 22'' \end{array}\right.$$

VI.

324. Given the three angles, *A*, *B*, and *C*.

The geometrical construction may be made by first drawing a triangle with its three sides equal to $180° - A$, $180° - B$, and $180° - C$, which is done in the way described in Art. 319, and then drawing the polar of this triangle. This polar triangle is the triangle required, for its angles, being supplements of the sides of the first triangle, are equal to *A*, *B*, and *C*.

325. FIRST METHOD. By the fundamental equations [155],

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

$$\cos b = \frac{\cos B + \cos C \cos A}{\sin C \sin A}$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}$$

Each of these equations gives two values of a side, one less and one greater than 180°, and such that

$$a_1 + a_2 = b_1 + b_2 + c_1 + c_2 = 360°.$$

326. **Example.**

Given, one angle = 125°, another = 130°, and another = 165°.

By the Table (p. 9).

	sin	cos
$A = 125°$	0.819	-0.574
$B = 130°$	0.766	-0.643
$C = 165°$	0.259	-0.966

By substitution in the equations,

$$\cos a = \frac{(-0.574) + (-0.643)(-0.966)}{(0.766)(0.259)} = 0.238$$

$$\cos b = \frac{(-0.643) + (-0.966)(-0.574)}{(0.259)(0.819)} = -0.417$$

$$\cos c = \frac{(-0.966) + (-0.574)(-0.643)}{(0.819)(0.766)} = -0.9415$$

$a_1 = 76° \ 15'$	$b_1 = 114° \ 38'$	$c_1 = 162° \ 6'$
$a_2 = 283° \ 45'$	$b_2 = 245° \ 22'$	$c_2 = 197° \ 54'$

327. SECOND METHOD. The halves of the sides can be computed from their sines [167], or from their cosines [169], or from their cotangents [170]. The most convenient form of computation from the cotangents is based on [176] and [177], namely,

$$\cot R = \sqrt{\frac{\cos(S-A)\cos(S-B)\cos(S-C)}{-\cos S}},$$

$$\cot \tfrac{1}{2} a = \frac{\cot R}{\cos(S-A)},\ \cot \tfrac{1}{2} b = \frac{\cot R}{\cos(S-B)},\ \cot \tfrac{1}{2} c = \frac{\cot R}{\cot S-C)},$$

The right-hand members of these equations, being square roots, are to be taken as both positive and negative. Hence each half side has two values, which doubled are the two values of the whole side.

328. Examples.

1. The same values as in Art. 326.

$A = 125°$			
$B = 130°$			
$C = 165°$			
$420°$			
$S = 210°$	log (−sec)	0.06247	(1)
$S-A = 85°$	log cos	8.94030	(2)
$S-B = 80°$	log cos	9.23967	(3)
$S-C = 45°$	log cos	9.84949	(4)
		18.09193	(5)
	log cot R =	9.04596	(6) = ½ (5)
$\frac{1}{2} a_1 = 38°\ 5'\ 53''$	log cot	10.10566	(7) = (6) − (2)
$\frac{1}{2} b_1 = 57°\ 22'\ 28''$	log cot	9.80629	(8) = (6) − (3)
$\frac{1}{2} c_1 = 81°\ 3'\ 57''$	log cot	9.19647	(9) = (6) − (4)

Taking the cotangents as negative,

$$\tfrac{1}{2}a_2 = 141^\circ\ 54'\ 7''$$
$$\tfrac{1}{2}b_2 = 122^\circ\ 37'\ 32''$$
$$\tfrac{1}{2}c_2 = 98^\circ\ 56'\ 3''$$

$$a_1 = 76^\circ\ 11'\ 46'' \qquad a_2 = 283^\circ\ 48'\ 14''$$
$$b_1 = 114^\circ\ 44'\ 56'' \qquad b_2 = 245^\circ\ 15'\ 4''$$
$$c_1 = 162^\circ\ 7'\ 54'' \qquad c_2 = 197^\circ\ 52'\ 6''$$

2. Given, each of the angles = 120°, to find the sides.

Results,

$$a_1 = b_1 = c_1 = 109^\circ\ 28'\ 17''$$
$$a_2 = b_2 = c_2 = 250^\circ\ 31'\ 43''$$

3. From the angles found in Art. 323, Ex. 2, compute the sides.

Results,

$$\left.\begin{aligned} a &= 109^\circ\ 33'\ 9'' \\ b &= 121^\circ\ 44'\ 6'' \\ c &= 53^\circ\ 34'\ 58'' \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} a &= 250^\circ\ 26'\ 51'' \\ b &= 238^\circ\ 15'\ 54'' \\ c &= 306^\circ\ 25'\ 2'' \end{aligned}\right.$$

4. Suppose a sphere to be circumscribed about a regular icosahedron, the points where the vertices of the latter touch the surface of the former being joined by great-circle arcs, so that the whole surface of the sphere is divided into twenty equal spherical triangles. These triangles are both equilateral and equiangular; and, since there are five triangles around each point, each angle measures 72°. What is the measure of each side?

Ans. 63° 26′ 4″.

CHAPTER XI.

THE GENERAL SPHERICAL TRIANGLE.

329. The general spherical triangle is one whose sides and angles are unlimited in value. Each one of its parts may exceed 180°, or 360°, or any number of times 360°. Since, however, all the functions of an arc or of an angle remain the same both in name and in value when that arc or angle is increased or diminished by any number of times 360°, there will be no loss of generality if the treatment of the general spherical triangle is limited to those forms of it in which each part is between 0° and 360° in value. The definition of a spherical triangle given in Art. 219, defines the general spherical triangle; but the limitation applied in the last part of that Article is now set aside, and all the different forms included under the definition are here considered.

330. There are in all eight different forms of the general spherical triangle, which may be enumerated and described as follows:

Forms 1 and 2.—Two forms (Fig. 75) in which every side is less than a semi-circumference. The first has every angle less than 180°, and is the ordinary form of spherical triangle heretofore considered. The second has the same sides as the first, but its angles are each greater than 180°, being equal to $360° - A$, $360° - B$, and $360° - C$ respectively. The two triangles together cover the whole surface of the sphere.

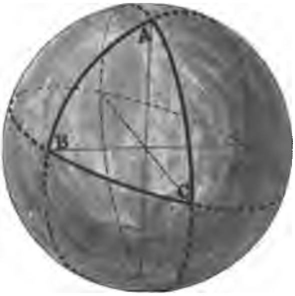


Fig. 75.

Forms 3 **and** 4.—Two forms (Fig. 76) in which two sides are less and one side greater than a semi-circumference. If A, B, and C denote the angles of Form 1, each angle being less than 180°, the angles of Form 3 are equal to $180° - A$, $180° - B$, and $360° - C$ respectively; and the angles of Form 4 are equal to $180° + A$, $180° + B$, and C. The two triangles together cover the whole surface of the sphere, one being the surface of a hemisphere with an ordinary triangle of Form 1 cut out of it, and the other the surface of the other hemisphere with the same triangle joined to it.

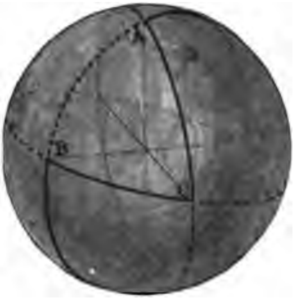


Fig. 76.

Forms 5 and 6.—Two forms (Fig. 77) in which one side is less and two sides are greater than a semi-circumference. The two greater sides cross each other at a point diametrically opposite the vertex of the angle they form. The angles of Form 5 are equal to A (vertically opposite), $180° + B$, and $180° + C$ respectively; while the angles of Form 6 are equal to $360° - A$, $180° - B$ and $180° - C$ respectively.

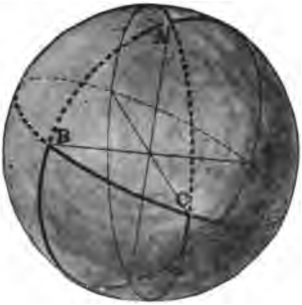


Fig. 77.

Forms 7 and 8.—Two forms (Fig. 78) in which every side is greater than a semi-circumference. Each side crosses another at a point diametrically opposite the vertex of the angle the two sides form. The angles of Form 7 are equal to $360^\circ - A$, $360^\circ - B$, and $360^\circ - C$ respectively; while the angles of Form 8, being vertically opposite those of the ordinary triangle, are equal to A, B, and C respectively.

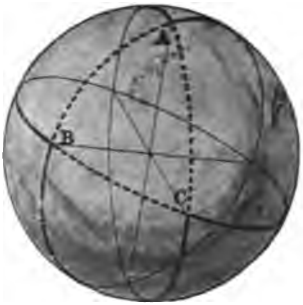


Fig. 78.

331. Using a, b, c, A, B, and C to denote the sides and angles of the ordinary spherical triangle, Form 1, the sides and angles of the different forms have the values given in the following

SUMMARY.

Form 1.		Form 2.	
a	A	a	$360^\circ - A$
b	B	b	$360^\circ - B$
c	C	c	$360^\circ - C$
Form 3.		**Form 4.**	
a	$180^\circ - A$	a	$180^\circ + A$
b	$180^\circ - B$	b	$180^\circ + B$
$360^\circ - c$	$360^\circ - C$	$360^\circ - c$	C
Form 5.		**Form 6.**	
a	A	a	$360^\circ - A$
$360^\circ - b$	$180^\circ + B$	$360^\circ - b$	$180^\circ - B$
$360^\circ - c$	$180^\circ + C$	$360^\circ - c$	$180^\circ - C$
Form 7.		**Form 8.**	
$360^\circ - a$	$360^\circ - A$	$360^\circ - a$	A
$360^\circ - b$	$360^\circ - B$	$360^\circ - b$	B
$360^\circ - c$	$360^\circ - C$	$360^\circ - c$	C

332. Some of the forms above described have occurred among the solutions of right and oblique spherical triangles in the foregoing chapters; but there they could be regarded only as particular results obtained from equations of limited generality. We shall now prove that the fundamental equations and all others derived from them are applicable to the general spherical triangle in all its forms, so that all limitations of the data as well as of the results of problems may be removed.

333. To prove that the fundamental equations

$$\cos a=\cos b\cos c+\sin b\sin c\cos A$$
$$\cos b=\cos c\cos a+\sin c\sin a\cos B$$
$$\cos c=\cos a\cos b+\sin a\sin b\cos C$$

are of unlimited generality.

Since $\cos(360^\circ-A)=\cos A$, no alteration in the value of the last term of the first equation is made by changing A to $360^\circ-A$. For the same reason, B may be changed to $360^\circ-B$, and C to $360^\circ-C$. Therefore the equations are applicable to spherical triangles of Form 2.

If the sides and angles of Form 3 are written in the equations, the last term of the first equation has two factors changed from positive to negative, because $\sin(360^\circ-c)=-\sin c$ and $\cos(180^\circ-A)=-\cos A$; but the term itself is unchanged either in sign or in value, so that the equation is undisturbed. The same is true of the second equation. The third equation is undisturbed, since $\cos(360^\circ-c)=\cos c$ and $\cos(360^\circ-C)=\cos C$. The equations, therefore, are applicable to spherical triangles of Form 3.

By similar reasoning it may be proved that the equations are applicable to spherical triangles of all the other forms. The equations, therefore, are of unlimited generality; and the same is true of all the equations and of all the methods of solution derived from them.

334. It should be said that equations [161], [163], [164], [167], [169], [170], [171], and [172] are perfectly general only when written with the double sign ± before the second member; but this circumstance does not disturb their use in computation. During computation the double sign may be disregarded; but at the end, when angles are to be taken out of the table, it must be taken into account.

335. The modulus, or the common value of the three ratios.

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}$$

is positive for triangles of Forms 1, 3, 5, and 7; because each side is like the opposite angle in respect of being less or greater than 180°; but for triangles of Forms 2, 4, 6, and 8 the modulus is negative. This statement is the guide in deciding which results of computation belong to one triangle, and which to the other.

SOLUTIONS OF THE GENERAL SPHERICAL TRIANGLE.

336. The methods of solution already explained are applicable to the general spherical triangle. The fundamental equations with three-place natural functions may be used when roughly approximate results are sought, and the object is merely to determine the shape of the required triangle. The simplicity and directness of this method make it a good safeguard against error in the use of more complicated methods. For logarithmic computation in cases where two sides and one angle or two angles and one side are given, the method by the Gaussian equations are preferable. Some points of detail in the application of logarithmic solutions are illustrated by the following examples.

337. *Given two sides and the included angle.*

Examples.

1. Given $a=292^\circ\ 34'$, $b=131^\circ\ 18'$, and $C=297^\circ\ 40'$.

By the Gaussian equations, Arts. 287, 288,

$\frac{1}{2}(a+b)=211^\circ\ 56'$	log sin	$9.72340n$	(1)
	cos	$9.92874n$	(2)
$\frac{1}{2}(a-b)=\ 80^\circ\ 38'$	sin	9.99417	(3)
	cos	9.21153	(4)
$\frac{1}{2}C\ =148^\circ\ 50'$	sin	9.71393	(5)
	cos	$9.93230n$	(6)
$\log\sin\frac{1}{2}(A+B)+\log\cos\frac{1}{2}c=$		$9.14383n$	(7) = (4) + (6)
$\log\sin\frac{1}{2}(A-B)+\log\sin\frac{1}{2}c=$		$9.92647n$	(8) = (3) + (6)
$\log\cos\frac{1}{2}(A+B)+\log\cos\frac{1}{2}c=$		$9.64267n$	(9) = (2) + (5)
$\log\cos\frac{1}{2}(A-B)+\log\sin\frac{1}{2}c=$		$9.43733n$	(10) = (1) + (5)
$\frac{1}{2}(A+B)=197^\circ\ 35'\ 33''$ or $377^\circ\ 35'\ 33''$	tan	9.50116	(11) = (7) − (9)
$\frac{1}{2}(A-B)=\ 72^\circ\ \ 2'\ \ 8''$	tan	10.48914	(12) = (8) − (10)
$\log\cos\frac{1}{2}(A+B)$		$9.97920np$	(13)
$\log\cos\frac{1}{2}(A-B)$		9.48915	(14)
		$9.66347pn$	(15) = (9) − (13)
		$9.94818n$	(16) = (10) − (14)
$\frac{1}{2}c=117^\circ\ 26'\ 10''$ or $62^\circ\ 33'\ 50''$	tan	$10.28471np$	(17) = (16) − (15)

$A_1=269^\circ\ 37'\ 41''$	$A_2=\ 89^\circ\ 37'\ 41''$
$B_1=125^\circ\ 33'\ 25''$	$B_2=305^\circ\ 33'\ 25''$
$c_1\ =234^\circ\ 52'\ 20''$	$c_2\ =125^\circ\ \ 7'\ 40''$

The value of $\frac{1}{2}(A+B)$, which must be greater than that of $\frac{1}{2}(A-B)$, is taken in the third and then in the fifth quadrant. The value of A_2, found by adding the value of $\frac{1}{2}(A-B)$ to the second value of $\frac{1}{2}(A+B)$, is $449^\circ\ 35'\ 33''$; but this may be diminished by 360°. The first triangle is of Form 5, the modulus being positive; the second is of Form 4, the modulus being negative.

2. Given,	Results,	
$a=123^\circ\ 15'$	$A_1=54^\circ\ 10'\ 18''$	$A_2=234^\circ\ 10'\ 18''$
$b=105^\circ\ 45'$	$B_1=68^\circ\ 55'\ 22''$	$B_2=248^\circ\ 55'\ 22''$
$C=255^\circ\ 30'$	$c_1=266^\circ\ 58'\ 48''$	$c_2=93^\circ\ 1'\ 12''$
	Form 3.	Form 2.

3. Given,	Results,	
$a=212^\circ\ 33'$	$A_1=218^\circ\ 10'\ 3''$	$A_2=38^\circ\ 10'\ 3''$
$b=121^\circ\ 51'$	$B_1=102^\circ\ 40'\ 29''$	$B_2=282^\circ\ 40'\ 29''$
$C=127^\circ\ 28'$	$c_1=43^\circ\ 42'\ 44''$	$c_2=316^\circ\ 17'\ 16''$
	Form 3.	Form 6.

4. Given,	Results,	
$a=302^\circ\ 50'$	$A_1=212^\circ\ 30'\ 45''$	$A_2=32^\circ\ 30'\ 45''$
$b=258^\circ\ 28'$	$B_1=218^\circ\ 48'\ 39''$	$B_2=38^\circ\ 48'\ 39''$
$C=163^\circ\ 42'$	$c_1=153^\circ\ 58'\ 30''$	$c_2=206^\circ\ 1'\ 30''$
	Form 5.	Form 8.

5. Given,	Results,	
$a=332^\circ\ 18'$	$A_1=333^\circ\ 23'\ 6''$	$A_2=153^\circ\ 23'\ 6''$
$b=246^\circ\ 8'$	$B_1=241^\circ\ 48'\ 14''$	$B_2=61^\circ\ 48'\ 14''$
$C=289^\circ\ 44'$	$c_1=257^\circ\ 36'\ 4''$	$c_2=102^\circ\ 23'\ 56''$
	Form 7.	Form 6.

338. *Given two angles and the included side.*

Examples.

1. Given, $A=117^\circ\ 20'$, $B=101^\circ\ 40'$, and $c=247^\circ\ 28'$.

By the Gaussian equations, Arts. 296, 297.

$\frac{1}{2}(A+B)=109° 30'$	log sin	9.97435	(1)
	cos	9.52350n	(2)
$\frac{1}{2}(A-B)= 7° 50'$	sin	9.13447	(3)
	cos	9.99593	(4)
$\frac{1}{2}c=123° 44'$	sin	9.91993	(5)
	cos	9.74455n	(6)
$\log\cos\frac{1}{2}(a-b)+\log\cos\frac{1}{2}C=$		9.71890n	(7) = (1) + (6)
$\log\sin\frac{1}{2}(a-b)+\log\cos\frac{1}{2}C=$		9.05440	(8) = (3) + (5)
$\log\cos\frac{1}{2}(a+b)+\log\sin\frac{1}{2}C=$		9.26805	(9) = (2) + (6)
$\log\sin\frac{1}{2}(a+b)+\log\sin\frac{1}{2}C=$		9.91586	(10) = (4) + (5)
$\frac{1}{2}(a-b)=167° 46' 58''$	tan	9.33550n	(11) = (8) − (7)
$\frac{1}{2}(a+b)=257° 19' 10''$ or 437° 19′ 10″	tan	10.64781	(12) = (10) − (9)
$\log\cos\frac{1}{2}(a-b)$		9.99005n	(13)
$\log\cos\frac{1}{2}(a+b)$		9.34147np	(14)
		9.72885	(15) = (7) − (13)
		9.92658np	(16) = (9) − (14)
$\frac{1}{2}C=122° 23' 9''$ or 57° 36′ 51″	tan	10.19773np	(17) = (16) − (15)

$a_1 = 65° 6' 8''$	$a_2 = 245° 6' 8''$
$b_1 = 89° 32' 12''$	$b_2 = 269° 32' 12''$
$C_1 = 244° 46' 18''$	$C_2 = 115° 13' 42''$
Form 3.	Form 8.

2 Given,	Results,	
$A=295° 41'$	$a_1 = 293° 11' 11''$	$a_2 = 113° 11' 11''$
$B=102° 29'$	$b_1 = 95° 11' 53''$	$b_2 = 275° 11' 53''$
$c = 128° 38'$	$C_1 = 49° 58' 55''$	$C_2 = 310° 1' 5''$
	Form 3.	Form 4.

3. Given,	Results,	
$A=214° 24'$	$a_1 = 320° 12' 11''$	$a_2 = 140° 12' 11''$
$B=137° 18'$	$b_1 = 129° 47' 53''$	$b_2 = 309° 47' 53''$
$c = 256° 12'$	$C_1 = 239° 0' 16''$	$C_2 = 120° 59' 44''$
	Form 5.	Form 6.

4. Given,

$A=309° 19'$	$a_1=229° 49' 4''$	$a_2=49° 49' 4''$
$B=264° 47'$	$b_1=259° 33' 18''$	$b_2=79° 33' 18''$
$c=63° 38'$	$C_1=69° 27' 47''$	$C_2=290° 32' 13''$
	Form 5.	Form 2.

5. Given,

$A=251° 8'$	$a_1=218° 33' 17''$	$a_2=38° 33' 17''$
$B=232° 22'$	$b_1=211° 26' 25''$	$b_2=31° 26' 25''$
$c=322° 52'$	$C_1=293° 34' 35''$	$C_2=66° 25' 25''$
	Form 7.	Form 4.

339. *Given two sides and the angle opposite one of them.*

Examples.

1. Given, $a=127° 25'$, $b=109° 27'$, and $A=234° 28'$.

By the Gaussian equations. Arts. 305, 306.

$a=127° 25'$	log csc 0.10005	(1)
$b=109° 27'$	sin 9.97448	(2)
$A=234° 28'$	sin 9.91051n	(3)
$B_1=255° 2' 45''$	sin 9.98504n	(4)=(1)+(2)+(3)
$B_2=284° 57' 15''$		
$\frac{1}{2}(A+B_1)=244° 45' 22'.5$	sin 9.95641n	(5)
	cos 9.62989n	(6)
$\frac{1}{2}(A-B_1)=-(10° 17' 22''.5)$	sin 9.25193n	(7)
	cos 9.99296	(8)
$\frac{1}{2}(a+b)=118° 26'$	sin 9.94217	(9)
	cos 9.67773n	(10)
$\frac{1}{2}(a-b)=8° 59'$	sin 9.19353	(11)
	cos 9.99464	(12)
$\log\cos\frac{1}{2}C_1-\log\cos\frac{1}{2}c_1=$	9.96177n	(13)= (5) −(12)
$\log\cos\frac{1}{2}C_1-\log\sin\frac{1}{2}c_1=$	0.05840n	(14)= (7) −(11)
$\log\sin\frac{1}{2}C_1-\log\cos\frac{1}{2}c_1=$	9.95216	(15)= (6) −(10)
$\log\sin\frac{1}{2}C_1-\log\sin\frac{1}{2}c_1=$	0.04879	(16)= (8) − (9)
$\frac{1}{2}c_1=38° 40' 39''.5$	tan 9.90337	(17)=(13) −(14)
	37	=(15) −(16)
$\frac{1}{2}C_1=135° 38' 2''.3$	tan 9.99039n	(18)=(15) −(13)
	39n	=(16) −14)

$\log\cos\frac{1}{2}C_2-\log\cos\frac{1}{2}c_2=$	$9.99832n$	$(19)=(8)n-(12)$
$\log\cos\frac{1}{2}C_2-\log\sin\frac{1}{2}c_2=$	$0.43636n$	$(20)=(6)-(11)$
$\log\sin\frac{1}{2}C_2-\log\cos\frac{1}{2}c_2=$	9.57420	$(21)=(7)-(10)$
$\log\sin\frac{1}{2}C_2-\log\sin\frac{1}{2}c_2=$	0.01224	$(22)=(5)n-(9)$
$\frac{1}{2}c_2=20^\circ\ 2'\ 17''$	tan 9.56196	$(23)=(19)-(20)$
$\frac{1}{2}C_2=159^\circ\ 21'\ 49''$	tan $9.57588n$	$(24)=(21)-(19)$

$c_1=77^\circ\ 21'\ 19''$ $\qquad c_2=40^\circ\ 4'\ 34''$

$C_1=271^\circ\ 16'\ 5''$ $\qquad C_2=318^\circ\ 43'\ 38''$

When, as in this example, sin B is negative so that $B_1+B_2=540^\circ$, we have

$$\begin{aligned}\sin\tfrac{1}{2}(A+B_2)&=-\cos\tfrac{1}{2}(A-B_1)\\ \cos\tfrac{1}{2}(A+B_2)&=\sin\tfrac{1}{2}(A-B_1)\\ \sin\tfrac{1}{2}(A-B_2)&=\cos\tfrac{1}{2}(A+B_1)\\ \cos\tfrac{1}{2}(A-B_2)&=-\sin\tfrac{1}{2}(A+B_1)\end{aligned}$$

so that (5) and (8), when used in the second part of the computation, must add or drop the *n*. But when $B_1+B_2=180^\circ$, it is (6) and (7) that are changed in that way. (See Art. 306.)

2. Given,	Results,	
$a=105^\circ\ 23'$	$B_1=189^\circ\ 16'\ 38''$	$B_2=350^\circ\ 43'\ 22''$
$b=331^\circ\ 21'$	$c_1=227^\circ\ 5'\ 26''$	$c_2=78^\circ\ 14'\ 43''$
$A=18^\circ\ 55'$	$C_1=345^\circ\ 44'\ 35''$	$C_2=160^\circ\ 46'\ 50''$
	Form 5.	Form 3.

3. Given,	Results,	
$a=102^\circ\ 35'$	$B_1=28^\circ\ 43'\ 16''$	$B_2=151^\circ\ 16'\ 44''$
$b=222^\circ\ 51'$	$c_1=318^\circ\ 10'\ 19''$	$c_2=109^\circ\ 36'\ 40''$
$A=316^\circ\ 24'$	$C_1=151^\circ\ 53'\ 9''$	$C_2=318^\circ\ 16'\ 16''$
	Form 6.	Form 4.

4. Given,	Results,	
$a=326^\circ\ 47'$	$B_1=38^\circ\ 18'\ 3'$	$B_2=141^\circ\ 41'\ 57''$
$b=106^\circ\ 9'$	$c_1=80^\circ\ 0'\ 14''$	$c_2=134^\circ\ 23'\ 50''$
$A=339^\circ\ 18'$	$C_1=140^\circ\ 32'\ 45''$	$C_2=27^\circ\ 27'\ 11''$
	Form 3.	Form 3.

5. Given,	Results,	
$a=208° 15'$	$B_1=205° 47' 22''$	$B_1=334° 12' 38''$
$b=110° 6'$	$c_1=43° 37' 21''$	$c_1=95° 15' 27''$
$A=167° 20'$	$C_1=198° 38' 23''$	$C_1=332° 31' 36''$
	Form 4.	Form 4.

6. Given,	Results,	
$a=285° 30'$	$B_1=209° 46' 38''$	$B_1=330° 13' 22''$
$b=215° 30'$	$c_1=230° 16' 56''$	$c_1=85° 43' 12''$
$A=235° 30'$	$C_1=221° 8' 3''$	$C_1=58° 31' 16''$
	Form 7.	Form 5.

7. Given,	Results,	
$a=219° 10'$	$B_1=76° 4' 40''$	$B_1=103° 55' 20''$
$b=299° 20'$	$c_1=242° 46' 8''$	$c_1=220° 35' 20''$
$A=135° 19'$	$C_1=98° 7' 14''$	$C_1=133° 34' 48''$
	Form 8.	Form 8.

340. *Given two angles, and the side opposite one of them.*

Examples.

1. Given $A = 75° 13'$, $B = 209° 23'$, and $a = 162° 36'$.

By the Gaussian equations, Arts. 314, 315.

$A=75° 13'$	log csc 0.01462	(1)
$B=209° 23'$	sin 9.69077*n*	(2)
$a=162° 36'$	sin 9.47573	(3)
$b_1=188° 43' 42''$	sin 9.18112*n*	(4)=(1)+(4)+(3)
$b_2=351° 16' 18''$		
$\frac{1}{2}(A+B)=142° 18'$	sin 9.78642	(5)
	cos 9.89830*n*	(6)
$\frac{1}{2}(A-B)=-(67° 5')$	sin 9.26429*n*	(7)
	cos 9.59039	(8)
$\frac{1}{2}(a+b_1)=175° 39' 51''$	sin 8.87854	(9)
	cos 9.99876*n*	(10)
$\frac{1}{2}(a-b_1)=-(13° 3' 51'')$	sin 9.35419*n*	(11)
	cos 9.98861	(12)

$\log\cos\frac{1}{2}C_1-\log\cos\frac{1}{2}c_1=$	9.79781	(13)= (5)−(12)
$\log\cos\frac{1}{2}C_1-\log\sin\frac{1}{2}c_1=$	0.61010	(14)= (7)−(11)
$\log\sin\frac{1}{2}C_1-\log\cos\frac{1}{2}c_1=$	9.89954	(15)= (6)−(10)
$\log\sin\frac{1}{2}C_1-\log\sin\frac{1}{2}c_1=$	0.71185	(16)= (8)− (9)
$\frac{1}{2}c_1=8^\circ\ 45'\ 31''$	tan 9.18771	(17)=(13)−(14)
	69	=(15)−(16)
$\frac{1}{2}C_1=51^\circ\ 39'\ 0''$	tan 10.10173	(18)=(15)−(13)
	75	=(16)−(14)
$\log\cos\frac{1}{2}C_2-\log\cos\frac{1}{2}c_2=$	0.90788*n*	(19)= (5)− (9)*n*
$\log\cos\frac{1}{2}C_2-\log\sin\frac{1}{2}c_2=$	9.96553	(20)= (7)−(10)
$\log\sin\frac{1}{2}C_2-\log\cos\frac{1}{2}c_2=$	0.54411	(21)= (6)−(11)
$\log\sin\frac{1}{2}C_2-\log\sin\frac{1}{2}c_2=$	9.60178*n*	(22)= (8)−(12)*n*
	tan 0.94235*n*	(23)=(19)−(20)
$\frac{1}{2}c_2=96^\circ\ 30'\ 54''$	33*n*	=(21)−(22)
	tan 9.63623*n*	(24)=(21)−(19)
$\frac{1}{2}C_1=156^\circ\ 35'\ 56''.5$	25*n*	=(22)−(20)

$c_1=17^\circ\ 31'\ 2''$	$c_2=193^\circ\ 1'\ 48''$
$C_1=103^\circ\ 18'\ 0''$	$C_2=313^\circ\ 11'\ 53''$
Form 3.	Form 5.

When, as in this example, sin b is negative, so that $b_1+b_2=540^\circ$ we have

$$\begin{aligned}\sin\tfrac{1}{2}(a+b_2)&=-\cos\tfrac{1}{2}(a-b_1)\\ \cos\tfrac{1}{2}(a+b_2)&=\sin\tfrac{1}{2}(a-b_1)\\ \sin\tfrac{1}{2}(a-b_2)&=\cos\tfrac{1}{2}(a+b_1)\\ \cos\tfrac{1}{2}(a-b_2)&=-\sin\tfrac{1}{2}(a+b_1)\end{aligned}$$

so that (9) and (12) when used in the second part of the computation must add or drop the *n*. But when $b_1+b_2=180^\circ$ it is (10) and (11) that are changed in that way. See Art. 315.

2. Given,	Results,	
$A=77^\circ\ 54'$	$b_1=21^\circ\ 24'\ 32''$	$b_2=158^\circ\ 35'\ 28''$
$B=211^\circ\ 48'$	$c_1=42^\circ\ 44'\ 16''$	$c_2=213^\circ\ 20'\ 34''$
$a=317^\circ\ 22'$	$C_1=258^\circ\ 26'\ 50''$	$C_2=52^\circ\ 30'\ 46''$
	Form 4.	Form 6.

3. Given,	Results,	
$A=216^\circ\ 10'$	$b_1=31^\circ\ 1'\ 18''$	$b_2=148^\circ\ 58'\ 42''$
$B=73^\circ\ 16'$	$c_1=328^\circ\ 35'\ 45''$	$c_2=200^\circ\ 23'\ 19''$
$a=341^\circ\ 29'$	$C_1=255^\circ\ 31'\ 38''$	$C_2=319^\circ\ 39'\ 23''$
	Form 5.	Form 5.

4. Given,	Results,	
$A=123^\circ\ 47'$	$b_1=235^\circ\ 43'\ 0''$	$b_2=304^\circ\ 17'\ 0''$
$B=71^\circ\ 53'$	$c_1=338^\circ\ 48'\ 8''$	$c_2=237^\circ\ 12'\ 24''$
$a=226^\circ\ 16'$	$C_1=24^\circ\ 34'\ 40''$	$C_2=104^\circ\ 46'\ 4''$
	Form 8.	Form 8.

5. Given,	Results,	
$A=207^\circ\ 56'$	$b_1=205^\circ\ 42'\ 14''$	$b_2=334^\circ\ 17'\ 46''$
$B=69^\circ\ 14'$	$c_1=332^\circ\ 26'\ 54''$	$c_2=198^\circ\ 31'\ 32''$
$a=167^\circ\ 27'$	$C_1=94^\circ\ 18'\ 24''$	$C_2=43^\circ\ 13'\ 56''$
	Form 6.	Form 6.

6. Given,	Results,	
$A=217^\circ\ 28'$	$b_1=214^\circ\ 7'\ 32''$	$b_2=325^\circ\ 52'\ 28''$
$B=298^\circ\ 30'$	$c_1=140^\circ\ 21'\ 20''$	$c_2=16^\circ\ 54'\ 24''$
$a=337^\circ\ 9'$	$C_1=91^\circ\ 53'\ 12''$	$C_2=27^\circ\ 6'\ 3''$
	Form 5.	Form 5.

7. Given,	Results,	
$A=213^\circ\ 25'$	$b_1=32^\circ\ 39'\ 50''$	$b_2=147^\circ\ 20'\ 10''$
$B=291^\circ\ 39'$	$c_1=144^\circ\ 44'\ 59''$	$c_2=21^\circ\ 3'\ 16''$
$a=161^\circ\ 21'$	$C_1=263^\circ\ 40'\ 56''$	$C_2=321^\circ\ 46'\ 49''$
	Form 2.	Form 2.

8. Given,	Results,	
$A=235^\circ\ 28'$	$b_1=216^\circ\ 5'\ 12''$	$b_2=323^\circ\ 54'\ 48''$
$B=210^\circ\ 16'$	$c_1=229^\circ\ 31'\ 20''$	$c_2=94^\circ\ 25'\ 19''$
$a=285^\circ\ 42'$	$C_1=220^\circ\ 36'\ 40''$	$C_2=121^\circ\ 26'\ 29''$
	Form 7.	Form 5.

341. *Given the three sides.*

Examples.

1. Given $a=97°$, $b=109°$, and $c=226°$.

By the method of Art. 322.

$a=$	$97°$		
$b=$	$109°$		
$c=$	$226°$		
$s=$	$216°$	log csc	$0.23078\,n$
$s-a=$	$119°$	sin	9.94182
$s-b=$	$107°$	sin	9.98060
$s-c=$	$-\ 10°$	sin	$9.23967\,n$
			19.39287
			$9.69643\frac{1}{2}\,p\,n$
$\frac{1}{2}A=$	$29°\ 36'\ 42.''4$	tan	$9.75461\frac{1}{2}\,p\,n$
$\frac{1}{2}B=$	$27°\ 27'\ 55''$	tan	$9.71583\frac{1}{2}\,p\,n$
$\frac{1}{2}C=$	$109°\ 15'\ 21''.3$	tan	$10.45676\frac{1}{2}\,n\,p$

$A_1=59°\ 13'\ 25''$	$A_2=300°\ 46'\ 35''$
$B_1=54°\ 55'\ 50''$	$B_2=305°\ 4'\ 10''$
$C_1=218°\ 30'\ 43''$	$C_2=141°\ 29'\ 17''$
Form 3.	Form 4.

log sin a, b, c,	9.99675	9.97567	$9.85693\,n$
log sin A, B, C,	$9.93408\,p\,n$	$9.91300\,p\,n$	$9.79427\,n\,p$
log modulus,	$0.06267\,p\,n$	$0.06267\,p\,n$	$0.06266\,p\,n$

2. Given,	Results,	
$a=84°$	$A_1=109°\ 4'\ 48''$	$A_2=250°\ 55'\ 12''$
$b=240°$	$B_1=304°\ 37'\ 4''$	$B_2=55°\ 22'\ 56''$
$c=230°$	$C_1=313°\ 17'\ 9''$	$C_2=46°\ 42'\ 51''$
	Form 5.	Form 6.

3.

Given,	Results,	
$a=320^\circ$	$A_1=312^\circ\ 5'\ 7''$	$A_2=47^\circ\ 54'\ 53''$
$b=310^\circ$	$B_1=297^\circ\ 48'\ 58''$	$B_2=62^\circ\ 11'\ 2''$
$c=300^\circ$	$C_1=270^\circ\ 53'\ 2''$	$C_2=89^\circ\ 6'\ 58''$
	Form 7.	Form 8.

4.

Given,	Results,	
$a=130^\circ$	$A_1=54^\circ\ 27'\ 54''$	$A_2=305^\circ\ 32'\ 6''$
$b=110^\circ$	$B_1=86^\circ\ 35'\ 19''$	$B_2=273^\circ\ 24'\ 41''$
$c=298^\circ$	$C_1=290^\circ\ 17'\ 22''$	$C_2=69^\circ\ 42'\ 38''$
	Form 3.	Form 4.

342. *Given the three angles.*

Examples.

1. Given $A=104^\circ$, $B=136^\circ$, and $C=264^\circ$.

By the method of Art. 327.

$A=104^\circ$		
$B=136^\circ$		
$C=264^\circ$		
$S=252^\circ$	log (−sec)	0.51002
$S-A=148^\circ$	cos	$9.92842\,n$
$S-B=116^\circ$	cos	$9.64184\,n$
$S-C=-\ 12^\circ$	cos	9.99040
		20.07068
		$10.03534\,n\,p$
$\frac{1}{2}a=38^\circ\ 1'\ 2''$	cot	$10.10692\,p\,n$
$\frac{1}{2}b=22^\circ\ 0'\ 15''$	cot	$10.39350\,p\,n$
$\frac{1}{2}c=137^\circ\ 57'\ 34''$	cot	$10.04494\,n\,p$

$a_1=76^\circ\ 2'\ 4''$	$a_2=283^\circ\ 57'\ 56''$
$b_1=44^\circ\ 0'\ 30''$	$b_2=315^\circ\ 59'\ 30''$
$c_1=275^\circ\ 55'\ 8''$	$c_2=84^\circ\ 4'\ 52''$
Form 3.	Form 6.

2. Given,	Results,	
$A=120^\circ$	$a_1=118^\circ\ 25'\ 58''$	$a_2=241^\circ\ 34'\ 2''$
$B=280^\circ$	$b_1=269^\circ\ 48'\ 51''$	$b_2=90^\circ\ 11'\ 9'$
$C=290^\circ$	$c_1=287^\circ\ 24'\ 34''$	$c_2=72^\circ\ 35'\ 26'$
	Form 5.	**Form 4.**

3. Given,	Results,	
$A=250^\circ$	$a_1=264^\circ\ 24'\ 14''$	$a_2=95^\circ\ 35'\ 46''$
$B=290^\circ$	$b_1=275^\circ\ 35'\ 46''$	$b_2=84^\circ\ 24'\ 14''$
$C=330^\circ$	$c_1=328^\circ\ 1'\ 28''$	$c_2=31^\circ\ 58'\ 32''$
	Form 7.	Form 2.

APPENDIX.

A.

Note to page 8.

THE arc-measure of an angle is the ratio to the radius of an arc drawn between the sides of the angle with the vertex for a centre. When, as is generally assumed, the length of the radius is a unit, the length of the arc itself is the arc-measure. The arc-measure of 360° is the ratio of the circumference to the radius, or 2π. The arc-measure of 180° is π; of 90°, $\frac{\pi}{2}$; of 45°, $\frac{\pi}{4}$; of 1°, $\frac{\pi}{180}$; of 1′, $\frac{\pi}{10080}$; of 1″, $\frac{\pi}{648000}$; of x degrees, $\frac{\pi x}{180}$; and so on.

The sine and the cosine of an angle are computed from the arc-measure, a, by means of the infinite series

$$\sin a = a - \frac{a^3}{1\times2\times3} + \frac{a^5}{1\times2\times3\times4\times5} - \frac{a^7}{1\times2\times3\times4\times5\times6\times7} + \cdots$$

$$\cos a = 1 - \frac{a^2}{1\times2} + \frac{a^4}{1\times2\times3\times4} - \frac{a^6}{1\times2\times3\times4\times5\times6} + \cdots$$

the proof of which may be found in books on the Differential Calculus.

The method of using these series is shown by the following examples.

(i) To find the sine of 23°.

$$\text{Arc-measure } 23^\circ=\frac{23\pi}{180}=0.40143$$

$$\begin{array}{lrr}
a=0.40143 & a= & 0.40143\\
a^3=0.06469 & -\dfrac{a^3}{1\times2\times3}= & -\underline{0.01078}\\
 & & 0.39065\\
a^5=0.01042 & +\dfrac{a^5}{1\times2\times3\times4\times5}= & \underline{0.00009}\\
 & & 0.39074\\
a^7=0.00168 & -\dfrac{a^7}{1\times2\times3\times4\times5\times6\times7}= & -\underline{0.00000}\\
 & & 0.39074
\end{array}$$

The remaining terms are too small to affect the result in the first five decimal places. The result is correct to four places at least; the fifth figure may be slightly erroneous. Be that as it may, it is certain that the correct three-place value for the sine of 23° is 0.391.

(ii) To compute the sine of 85°.

$$\text{Arc-measure } 85^\circ=\frac{85\pi}{180}=1.48355$$

$$\begin{array}{lrr}
a=1.48355, & & a=1.48355\\
a^3=3.26514, & -\dfrac{a^3}{1\times2\times3}= & -\ \underline{0.54419}\\
 & & 0.93936\\
a^5=7.1862, & +\dfrac{a^5}{1\times2\times3\times4\times5}= & +\ \underline{0.05988}\\
 & & 0.99924\\
a^7=15.816, & -\dfrac{a^7}{1\times2\times3\times4\times5\times6\times7}= & -\ \underline{0.00314}\\
 & & 0.99610\\
a^9=34.81, & +\dfrac{a^9}{1\times2\times3\times4\times5\times6\times7\times8\times9}= & +\ \underline{0.000096}\\
 & & 0.996196\\
a^{11}=76.61, & -\dfrac{a^{11}}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11}= & -\ \underline{0.000002}\\
 & & 0.996194
\end{array}$$

$$\sin 85^\circ=0.996$$

(iii) To compute the cosine of 25°.

$$\text{Arc-measure } 25^\circ = \frac{25\pi}{180} = 0.43633$$

$$\begin{array}{lrr}
 & & 1.00000 \\
a^2 = 0.19039, & -\dfrac{a^2}{1\times 2} = & -\underline{0.09519} \\
 & & 0.90481 \\
a^4 = 0.03625, & +\dfrac{a^4}{1\times 2\times 3\times 4} = & +\underline{0.00151} \\
 & & 0.90632 \\
a^6 = 0.00690, & -\dfrac{a^6}{1\times 2\times 3\times 4\times 5\times 6} = & -\underline{0.00001} \\
 & & 0.90631 \\
a^8 = 0.00131, & +\dfrac{a^8}{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8} = & +\underline{0.00000} \\
 & & 0.90631
\end{array}$$

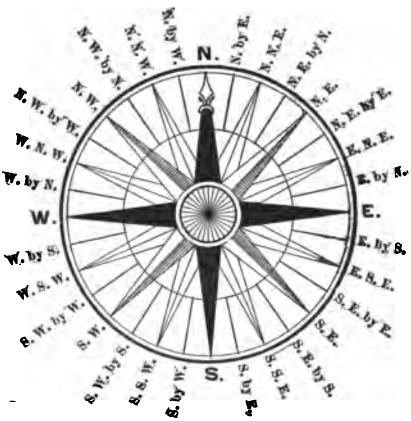
$$\cos 25^\circ = 0.906$$

In the same way may be computed all the values in the first and last columns of Table I (p. 9).

For finding the successive powers of a, logarithms may be used to advantage.

B.

THE MARINER'S COMPASS.



THE mariner's compass consists of a magnetic needle carrying a circular card so suspended that the card keeps a horizontal position with the north pole of the needle always directed towards the north point of the horizon. Thus the card represents the plane of the horizon, and the radial lines marked upon it (see Figure) show the directions of points around the horizon.

The cardinal points, north (N.), south (S.), east (E.), and west (W.) divide the horizon into four quadrants of 90° each.

Each quadrant is halved by the points northeast (N. E.), southeast (S. E.), southwest (S.W.), and northwest (N.W.).

Each half-quadrant is halved by the points north-northeast (N. N. E.), east-northeast (E. N. E.), east-southeast, (E.S.E.), south-southeast (S.S.E.), south-southwest (S.S.W.), west-southwest (W. S.W.), west-northwest (W. N.W.), and north-northwest (N. N. W.).

Finally, the points north by east (N. by E.), northeast by north (N. E. by N.), northeast by east (N. E. by E.), east by north (E. by N.), and so on around the circle, divide each quarter of a quadrant into two equal parts.

Thus there are thirty-two points of the compass. Naming these points in succession around the circle* is called *boxing the compass.*

The word *point* is also used to denote the angle between two radial lines drawn to any two successive points of the compass.

Thus the angle between the lines marked E. S. E. and S. E. by E. is an angle of *one point,* equal to one-eighth part of a right angle, or 11° 15′. Again the angle between the lines marked N. and N. E. by N. is an angle of three points, equal to 33° 45′.

For greater precision in recording courses and bearings, points are divided into halves and quarters. Thus a direction half-way between W. N. W. and W. by N. is designated W. N. W. ½ W.; a direction 6¼ points to the east of north is designated E. N. E. ¼ E. The angle between these two directions is an angle of 12¾ points, or 143° 26′ 15″.

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* Thus: north, north by east, north-northeast, northeast by north, northeast, northeast by east, and so on; or north, north by west, north northwest and so on.

C.

Note to Article 208, page 155.

The following is a more direct way of deriving the formulas [138] and [139] than that given in the text:

By [63] and (64],

$$\sin (B-A)=\sin B \cos A-\cos B \sin A$$
$$\cos (B-A)=\cos B \cos A+\sin B \sin A$$

But

$$\sin A=\cos B=\frac{a}{h}$$
$$\cos A=\sin B=\frac{b}{h}$$

Therefore, by substitution,

$$\sin (B-A)=\frac{b^2}{h^2}-\frac{a^2}{h^2}=\frac{(b+a)(b-a)}{h^2}$$
$$\cos (B-A)=\frac{a\,b}{h^2}+\frac{a\,b}{h^2}=\frac{2\,a\,b}{h^2}$$

ANSWERS.

Article 24. **Page 15.**

Ex. 1. $\sin = 0.600 \quad \cos = 0.800$
$\tan = 0.750 \quad \cot = 1.333$
$\sec = 1.250 \quad \csc = 1.667$

2. $\sin = 0.923 \quad \cos = 0.385$
$\tan = 2.400 \quad \cot = 0.417$
$\sec = 2.600 \quad \csc = 1.083$

3, 4, 5. See Table I, page 9.

6. $\cos A = \frac{1}{3}\sqrt{5} = 0.745$
$\tan A = \frac{2}{5}\sqrt{5} = 0.894$
$\cot A = \frac{1}{2}\sqrt{5} = 1.118$
$\sec A = \frac{3}{5}\sqrt{5} = 1.342$
$\csc A = \frac{3}{2} = 1.500$

7. $\sin A = \sqrt{\frac{3}{5}} = 0.775$
$\tan A = \sqrt{\frac{3}{2}} = 1.225$
$\cot A = \sqrt{\frac{2}{3}} = 0.816$
$\sec A = \sqrt{\frac{5}{2}} = 1.581$
$\csc A = \sqrt{\frac{5}{3}} = 1.291$

8. $\cot A = \frac{1}{3} = 0.333$
$\sec A = \sqrt{10} = 3.162$
$\cos A = \frac{1}{10}\sqrt{10} = 0.316$
$\sin A = \frac{3}{10}\sqrt{10} = 0.949$
$\csc A = \frac{1}{3}\sqrt{10} = 1.054$

9. $\tan A = \frac{4}{3} = 1.333$
$\sec A = \frac{5}{3} = 1.667$
$\csc A = \frac{5}{4} = 1.250$
$\sin A = \frac{4}{5} = 0.800$
$\cos A = \frac{3}{5} = 0.600$

10. $\cos A = \frac{1}{3} = 0.333$
$\sin A = \frac{2}{3}\sqrt{2} = 0.943$
$\csc A = \frac{3}{4}\sqrt{2} = 1.061$
$\tan A = 2\sqrt{2} = 2.828$
$\cot A = \frac{1}{4}\sqrt{2} = 0.354$

11. $\sin A = \frac{1}{2} = 0.500$
$\cos A = \frac{1}{2}\sqrt{3} = 0.866$
$\tan A = \frac{1}{3}\sqrt{3} = 0.577$
$\cot A = \sqrt{3} = 1.732$
$\sec A = \frac{2}{3}\sqrt{3} = 1.155$

12, 13. See Table I, page 9.

Article 24. **Page 16.**

16. 0.115
17. 0.958
18. 0.944
19. 0.673
20. 0.199
21. 2.396
22. 2.301
23. 0.770
24. 1.029
25. 2.534
26. 1.924
27. 1.017
28. 8° 21′
29. 55° 36′
30. 18° 36′
31. 84° 17′
32. 41° 51′
33. 36° 51′
34. 17° 28′
35. 61° 55′
36. 20° 20′
37. 30° 56′
38. 39° 13′
39. 35° 18′
40. 26° 33′
41. 33° 34′
42. 18° 27′
43. 15° 29′
44. 60°
45. 41° 50′

Article 37. **Page 21.**

Ex. 1. The other angle = 42° 24′; the sides, 11.81 and 10.78.

2. The other angle = 79° 46′; the sides 1 ft. 2.24 in., 6 ft. 6.72 in.

3. The other angle = 67° 5′; the base = 82.81 ft.; and the hypothenuse = 89.88 ft.

4. The other angle = 23° 18′; the perpendicular = 11 ft. 9.9 in.; and the hypothenuse = 29 ft. 11.4 in.

5. The other angle = 54° 45′; the base = 30.05 ft.; and the hypothenuse = 52.06 ft.

6. The other angle = 50° 15′; the perpendicular = 2.879 metres; and the hypothenuse = 4.498 metres.

7. Angles 29° 0′ and 61° 0′; the other side = 15.49 ft.

8. Angles 45° 45′ and 44° 15′; the other side = 4.67 inches.

9. Angles 22° 39′ and 67° 21′; the hypothenuse = 221 feet.

10. Angles 50° 17′ and 39° 43′; the hypothenuse = 0.5667.

11. 32.9 feet; 20° 4′.

12. 3° 25′.

13. 20.22 feet; 8° 32′.

Article 37. **Page 22.**

14. 2° 53′.

15. 14 feet.

16. With the longer side of the floor, 45° 28′; with the shorter side, 54° 15′; with the diagonal of the floor, 24° 8′.

17. 89.16 rods north; 80.28 rods east.

18. 133.8 rods north; 148.6 rods west.

19. North 42° east; 53.84 miles.

Page 23.

20. 60.75 rods east; walked 96.52 rods.

21. Sailed 40.9 miles; headland 22.7 miles distant.

22. 99.1 miles apart; south 50° 31′ west.

23. North 50° 32′ east; 18.73 rods distant.

24. 59.08 rods.

25. 47.25 miles per hour.

Page 24.

26. 138.4 feet.

27. 1415 feet.

28. 234.2 feet.

29. 1645 feet. By 5-place table, 1636.2 feet.

See remarks on the next answer.

30. By the three-place table 4372 feet. But the error in this result is considerable, because the variations of the cotangents of small angles are out of proportion to the variations of the angles themselves, making the method of simple interpolation inexact. The natural cotangent of 3° 20′ is 17.169 (see *Log. Trig. Tab.*, p. 142), but simple interpolation in Table I (p. 9), gives 17.49. The answer by the five-place table is 4292.3 feet.

31. 329 feet.

32. 6.93 miles an hour.

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33. 301 feet.

34. Street 126.6 feet wide. First window 15.95, second 34.94 feet above street.

35. 74.8 feet.

Article 37. **Page 26.**

36. 22.3 minutes, 4.01 miles off.

37. 29° 44′.

38. Latitude 33° 42′.

39. 39.21 feet.

40. 38° 0′, 71° 0′, and 71° 0′.

41. 40° 6′, 69° 57′, and 69° 57′.

42. Height 407 feet; each side 421 feet.

43. Base 29.6 feet; height 32.8 feet.

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44. 33° 57′.

45. Diameter 8.72 inches; slant height 20.46 inches.

46. Each face inclined 72° 39′; each edge, 66° 9′; each edge makes an angle of 33° 10′ with that in the same face with it, and an angle of 47° 42′ with that not in the same face.

47. The smaller acute angle is 30° [22° 30′, 18°, 15°, 12°, 9°].

48. 7.06 [4.60, 3.71, 3.11, 2.50, 1.87] inches.

49. 16.00 [11.63, 6.62, 5.20, 4.29, 2.53] inches.

50. 8.68 inches.

51. 3.08 feet.

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55. As the cosine of 18° is to 1, or as 951 is to 1000. For four sides, as 707 to 1000; five, as 809 to 1000; six, as 866 to 1000; eight, as 924 to 1000; twelve, as 966 to 1000; twenty, as 988 to 1000.

Article 54. **Page 34.**

Ex. 1. Sides, 12501 feet and 938.5 feet.

2. Perpendicular, 1738.3 feet; hypothenuse, 33666 feet.

3. Angle, 1° 6′ 43″; hypothenuse, 12625.3 feet.

4. Inclination, 22′.47″.5.

5. Angles, 4° 48′ 37″ and 85° 11′ 23″.

6. Chord, 467.4 feet.

Article 54. **Page 34.**

7. Height 877.5 feet; elevation at two miles 4° 45′ 1″.

8. Distance, 874.44 metres; chord, 118.02 metres.

9. Base, 105.06 feet.

10. Side, 140.57 metres.

11. Chord, 69.16 miles; between chord and surface, 0.15088 of a mile, or 796.6 feet.

12. Side, 302.7 metres; distance from centre to vertex, 348.8 metres.

Article 63. **Page 38.**

Ex. 1. Third angle, 69°; the other sides, 81.12 and 67.64 rods.

2. Third angle, 30° 58′; the other sides, 62.23 and 40.08 feet.

3. Third angle, 55° 2′; the other sides, 31.29 and 26.40 rods.

4. Third angle, 53° 50′; the other sides 14.39 and 13.12 feet.

5. Sides, 23.62 and 21.29 feet.

6. Sides, 7.178 and 6.902 inches.

7. 13.87 miles.

8. From one end 151.17 feet; from the other 128.81 feet.

9. 211.0 rods.

Article 69. **Page 41.**

1. Angles, 64° 35′ and 51° 8′; third side, 28.93 feet.

2. Angles, 26° 33′ and 29° 45′; third side, 68.80 rods.

3. Angles, 44° 0′ and 63° 20′; third side, 48.08.

4. Angles, 30° 20′ and 31° 15′; third side, 189.9.

5. Angles, 50° 2′ and 105° 40′; third side, 39.50.

6. Angles, 5° 27′ and 16° 17′; third side, 83.85.

7. Angles, 48° 32′ and 61° 49′; third side, 29.04.

8. Angles, 36° 56′ and 44° 16′; third side, 9.203.

9. Angles, 9° 7′ and 10° 53′; third side, 270.7.

Article 69. **Page 41.**

10. Angles, 48° 46′ and 75° 19′; third side, 10.79.

11. Angles, 68° 44′ and 78° 40′; third side, 10.99.

12. Angles, 11° 19′ and 11° 26′; third side, 197.1.

13. 17.79 miles.

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14. 133.0 rods.

15. 109.5 rods.

16. 38.37 miles.

17. 14.90 miles.

18. 18.30 inches.

19. A C = 161.2 rods, A C B = 44° 23′.
B D = 153.6 rods, D B C = 49° 51′.
A D = 82.4 rods.

20. 48.66 rods.

21. Sides, 165.9, 194.3, 135.0; angles 57° 10′, 79° 45′, 43° 5′.

Article 77. **Page 52.**

1. Third side, 22.10 or 5.29 feet.
Adjacent angle, 62° 8′ or 117° 52′.
Opposite angle, 68° 35′ or 12° 51′.

2. Third side, 38.26 or 22.00 feet.
Adjacent angle, 77° 18′ or 102° 42′.
Opposite angle, 52° 32′ or 27° 8′.

3. Third side, 101.36 or 26.72 rods.
Adjacent angle, 55° 8′ or 124° 52′.
Opposite angle, 85° 0′ or 15° 16′.

4. Third side, 24.04 or 6.63 inches.
Adjacent angle, 69° 37′ or 110° 23′.
Opposite angle, 53° 34′ or 12° 48′.

5. Third side, 34.95 or 7.05.
Adjacent angle, 80° 17′ or 99° 43′.
Opposite angle, 24° 11′ or 4° 45′.

6. Impossible; sin $B > 1$.

Article 77. **Page 52.**

7. Third side, 47.64.
Adjacent angle, 55° 0′.
Opposite angle, 52° 42′

8. Impossible.

9. Third side, 91.43.
Adjacent angle, 56° 4′.
Opposite angle, 55° 36′.

10. Third side, 7.07.
Adjacent angle, 78°.
Opposite angle, 24°.

11. Third side, 32.51 feet; adjacent angle, 90°; opposite angle, 52° 29′.

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12. Third side, 9.61 feet; adjacent angle, 47° 35′; opposite angle, 7° 48′.

13. Third side, 22.82 feet; adjacent angle, 30° 48′; opposite angle, 6° 36′.

14. Inadmissible.

15. Impossible.

16. Third side, 9.537 or 1.516 metres; adjacent angle, 41° 5′ or 138° 55′; opposite angle, 106° 37′ or 8° 47′.

17. Third side, 113 yards; adjacent angle, 3° 50′; opposite angle, 3° 55′.

18. Third side, 105.64 or 92.56 yards; adjacent angle, 64° 8′ or 115° 52′; opposite angle, 108° 7′ or 56° 23′.

19. Third side, 78.33 feet, adjacent angle, 29° 30′; opposite angle, 50° 30′.

20. Distances 5.86 and 21.32 inches.

Article 80. **Page 55.**

1. 54° 30′, 58° 32′, and 66° 58′.
2. 44° 25′, 57° 8′, and 78° 27′.
3. 14° 15′, 18° 24′, 147° 21′.
4. 43° 0′, 56° 34′, 80° 26′.

Article 80. **Page 55.**

5. 43° 0′, 56° 34′, 80° 26′.
6. 43° 0′, 56° 34′, 80° 26′.
7. 42° 45′, 53° 13′, 84° 2′.
8. 103° 31′.

Article 86. **Page 58.**

1. Third angle, 53° 54′ 56″; sides 4889.1 and 4534.1 feet.

2. Third angle, 31° 58′ 40″; sides, 1052.88 and 693.37 metres.

3. Third side, 57.693 or 3.651 feet; adjacent angle, 57° 10′ 35″ or 122° 49′ 25″; opposite angle, 69° 2′ 5″ or 3° 23′ 15″.

4. Third side, 52.372 feet; angles, 54° 34′ 49″ and 63° 14′ 46″.

5. 328.04 and 378.20 feet.

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6, Third side, 148.321; angle adjacent, 64° 47′ 30″; angle opposite, 26° 52′ 55″.

7. Angles, 50° 28′ 4″, 57° 54′ 27″, and 71° 37′ 29″.

8. Third side, 462.35 metres; angles, 45° 18′ 26″ and 45° 26′ 59″.

9. Third side, 460.53 yards; angle adjacent, 2° 12′ 13″ (found from its tangent); angle opposite, 2° 19′ 7″.

10. Angles, 48° 5′ 26″, 61° 57′ 57″, 69° 56′ 37″.

11. Third side, 1862.13 feet; angles, 2° 44′ 52″ and 5° 50″ 52″.

12. Angles, 28° 25′ 0″, 28° 46′ 20″, and 122° 48′ 40″.

13. Third side, 992.14 or 1.00 feet; angle adjacent, 2° 23′ 32″.5 (found from its tangent) or 177° 36′ 27″.5; angle opposite 175° 13′ 12″.5 or 0° 0′ 17″.5.

14. Angles, 59° 54′ 3″, 60° 0′ 0″, and 60° 5′ 57″.

15. Third side, 800.74; angles, 41° 49′ 33″ and 41° 52′ 2″.

16. Third side to first as 1.68123 to 1; to second as 1.68123 to 0.993465; angles 32° 22′ 50″ and 32° 37′ 10″.

Article 86. **Page 59.**

With seven-place tables the first term of each ratio is found to be 1.681275.

17. Angles, 55° 35′ 22″ Perpendiculars, 2982.7
58° 52′ 51″ 2874.3
65° 31′ 47″ 2703.5

18. Angle between the two roads, 50° 22′.

Article 87. **Page 60.**

1. 9.2448 inches.

2. 15.6252 inches.

3. 32.516 feet.

4. Height, 840.1 feet; distance, 1374.5.

5. Sides, 16.280 and 29.308 inches; angles, 47° 52′ 51″ and 132° 7′ 9″.

6. 19.009 feet.

Page 61.

7. Lengths, 42.720, 43.081, and 44.287.
Angles, 35° 16′ 33″, 37° 21′ 6″, and 43° 6′ 39″.

8. C D = 3.7571 miles.

9. E F = 4.7358 miles.

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10. Sailed 17.97 miles on a course N. 13° 58′ E.

11. 1026.52 feet apart. A distant 1157.47 feet, and B 931.84 feet from monument.

12. Sides, 1841.79, 1834.42, and 1200.53 feet.
Angles 71° 16′ 17″, 70° 36′ 28″, and 38° 7′ 15″.
Tower distant from A, 795.48 feet; from B, 1105.225 feet; from C, 971.18 feet.

13. 10.807 chains.

Page 63.

14. Angles of triangle 79° 43′ 14″, 42° 4′ 38″, and 58° 12′ 8″.
B D = 361.0, D C = 274.99.
O A = 1225.8, O B = 883.10, O C = 954.37.

Article 138. **Page**

1. +0.612
2. −0.568
3. −0.805
4. −0.408
5. +4.940
6. −2.028

7. −1.025
8. −1.061
9. +1.275
10. −0.221
11. −0.7715
12. +0.262

13. −1.627
14. +0.864
15. −0.29
16. +1.76
17. −1.17
18. −1.81

19. 21° 32′ and 158° 28′.
20. 244° 34′ and 295° 26′.
21. 57° 44′ and 237° 44′.
22. 150° 23′ and 330° 23′.
23. 32° 28′ and 327° 32′.
24. 151° 30′ and 208° 30′.

25. 65° 30′ and 294°
26. 113° 30′ and 246
27. 72° 32′ and 252°
28. 158° 12′ and 338
29. 13° 28′ and 166°
30. 207° 44′ and 332

31. 9.76636
32. 9.78371*n*
33. 9.88198*n*
34. 9.78252*n*
35. 9.87249*n*
36. 9.82044

37. 9.96467*n*
38. 10.01233
39. 10.01170*n*
40. 9.90457*n*
41. 10.19395
42. 9.67338*n*

43. 14° 41′ 41″ and 165° 18′ 19″.
44. 238° 16′ 23″ and 301° 43′ 37″.
45. 58° 14′ 25″ and 238° 14′ 25″.
46. 144° 11′ 24″ and 324° 11′ 24″.
47. 46° 18′ 36″ and 313° 41′ 24″.
48. 136° 2′ 50″ and 223° 57′ 10″.
49. 55° 47′ 11″ and 235° 47′ 11″.
50. 144° 48′ 44″ and 324° 48′ 44″.

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